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The Simpson's Sunblocker: Similarity and Congruence through Modeling, Exploration, and Reasoning

This unit was created by Jo Boaler and 3 of her graduate students for a local school - in alphabetical order: Dan Meyer, Sarah Kate Selling and Kathy Sun. We thank IMP and Mark Driscoll for allowing us to use problems from their books.

Modifications and updates were provided by *youcubed*.

The unit contains 19 lessons and two formal assessments.

Geogebra is used in this unit. Go to http://www.geogebra.org/ for a free download.

This unit addresses two essential questions:

1. How do we use mathematics to find the height of things that are difficult to measure?
2. How do we make a convincing argument?

The High School Common Core State Standards addressed are:

Geometry: Similarity, Right Triangles, and Trigonometry (G.SRT)1,2,3,4,5,6,8
Geometry: Congruence (G.CO) 7,8
Math Mathematical Practices (MPS) 1,2,3,4,5,6,7,8

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Design Principles

Part I: Student Engagement Principles

Goals for students:

- Engage in sense making at all times.
- See learning math as an activity that they need to *actively engage* with, rather than *passively receive*.
- Understand that mathematical ideas need to be discussed in order to develop understanding.
- See themselves engaged in collaborative problem solving — which means they need to:
  - Feel comfortable offering ideas that may be incorrect.
  - Listen to each other's ideas.
  - Respect different people's ideas, regardless of "status" or prior understanding.
- Develop persistence as a mathematical problem solver.
- See math learning as requiring different types of activity — for example, investigation, conjecturing, practice, discussion. (See below.)

Part II: Unit Design Principles

As we worked we:

- Determined the mathematical area.
- Decided on the essential questions.
- Determined the big ideas and the specific mathematics content and practices within them.
- Considered assessments.
- Chose, designed or adapted activities.

As we chose and designed activities, our goals were, whenever possible, to:

- Start with a question that students could access using what they knew (ie. they could start the activity without any direct instruction) so that work on the activity prompts the need for some new math knowledge.

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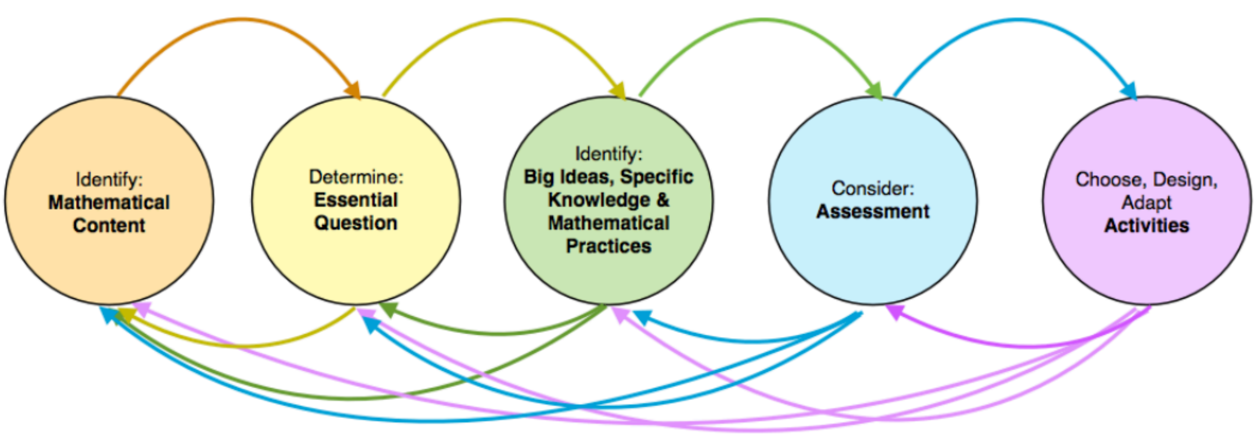
- Use activities that involve data from their lives.
- Allow students to *discover* mathematical concepts.
- Encourage students to use their intuition.
- Combine different types of activity — investigations, discussions, practice, individual and group work, reflections, poster presentations, mathematical writing, etc.
- Choose activities that are sufficiently open for students to access in different ways and at different levels. This is our model for differentiation. In some instances, we also included extensions.
- Make sure that students know what task they're working on at all times.
- Include appropriate scaffolding.
- Involve technology when it enhances the learning experience.

Part III: Pedagogical Practices in the Unit

- Teacher launching a problem
- Small group investigation – including making a conjecture
- Small group discussion
- Whole class debrief of investigation
- Poster presentations
- Exit tickets



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Day 1

Essential Question	Lesson Objective	Standards
How do we use math to find the height of things that are difficult to measure?	Students understand a proportional relationship between two variables.	G-SRT.1. Verify experimentally the properties of dilations given by a center and a scale factor MP.8. Look for and express regularity in repeated reasoning.

Activities

1. **Introduce the Essential Question.**

Slideshow	Set the scene for this unit of study. Show images of things whose height might be interesting to measure. Ask students to generate a list of objects whose height might be interesting to measure.

2. Establish Classroom Norms. Discussion around norms for working together and doing math. (ie. How to have a discussion, how to justify answers, etc.)
3. Simpsons Sunblocker. The purpose of this activity is to give students an opportunity to explore the concept of geometric proportionality. Since students will be working in teams, this activity helps to establish group norms.

Launch	Show the Simpson's clip of Mr. Burns blocking the sun with a circular disk. State the goal of the day: "Our goal is to help Mr.

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	Burns understand where to place a circular disk in order to block the sun over the town of Springfield." • Show the contents of the Sunblocker kit: a flashlight, the cut-out sun blockers, a ruler, and the map of the city of San Jose (or a city of your choice). • Ask several students to help demonstrate the data collection. One student holds the flashlight, another holds a blocker, and another measures. The fourth records. Point out that students need to keep the flashlight in the same position at all times, as much as possible, just like the sun is. • Explicitly call attention to what each member of the group is doing. • Distribute the Simpsons Sunblocker worksheet.

4. Simpsons Sunblocker

Group Work	In groups of four, students should repeat the experiment for all the Sunblockers in their kit. Students should create a table with the following variables: diameter of the blocking disk and distance from light source. Inform students they will be challenged at the end of the day by being given a disc they have not used. Their goal will be to put the disc in the best location to block out the sun over San Jose. Note: Exchange the map in the unit for any location that would be relevant and of interest to the students. Teacher Talking Points: • Precision. Encourage students to be as precise as possible in their measurements and in holding the flashlight. • Noticing. Ask students to share their informal observations about what they're noticing. Student Challenges & Misconceptions: Organization. Students might not experiment systematically. They might measure the discs in an order that doesn't highlight

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	patterns as well as ascending or descending. If they struggle to see patterns ask students how they could organize their data differently.
Formative Assessment	When groups have completed the table, pass out the next prompt on a card: "We've created a disc that is 15 cm in diameter. Where should it go? Everybody in your group should be able to explain your reasoning." Student Conceptions: • Estimation — Direction. "I know the blocker is going to be farther away from the light because the disk is bigger." Estimation — Constant Difference. "I know the blocker is going to be farther away and I used the pattern to count up to where I thought it would be." Estimation — Interpolation. "San Jose is 18.5 centimeters. The largest disk is 9.4. This new disk is 15 centimeters so it is going to be really far away from the light source." • Calculation. The student calculates the constant of proportionality and uses it to find the new height.
Group Work	As groups find their estimates, the teacher asks a random member for her answer and her reasoning. (The answer should be more sophisticated than Estimation — Direction.) Teacher passes out the two graphs. The first is to be used for a sketch. The second is to be used for a more accurate graph. Every student in the group should make a sketch of the data (rough, no units, to get a sense of trends) and a graph of the data (precise, using grid paper) and compare with each other. They will explain to each other in words why they drew it the way they did. Teacher Talking Points: • Methods. Teacher will ask students how they will use their graph to determine the height of the final sun blocker. Teacher can ask students, "Can you show me where the final sun blocker is on your graph?" Once students have their answer — both estimated and now tested by graphing — the teacher can test their answer by giving them the final sunblocker and letting them use it. Each group should record how close their answer was from the actual best placement of the final disk. (Note: every group may have a different, correct answer, depending on where they hold the flashlight.)Teacher will offer extensions as the groups finish. Every group needs to

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	enter their data in a Google Form (or equivalent) so the teacher can pose questions about the class data set the next day.
Class Debrief	Headlining. Each group decides on a single "headline," a major takeaway or big idea from today. (eg. "Use little data to make conclusions about big data.") They write it down and then share each out to the entire class.

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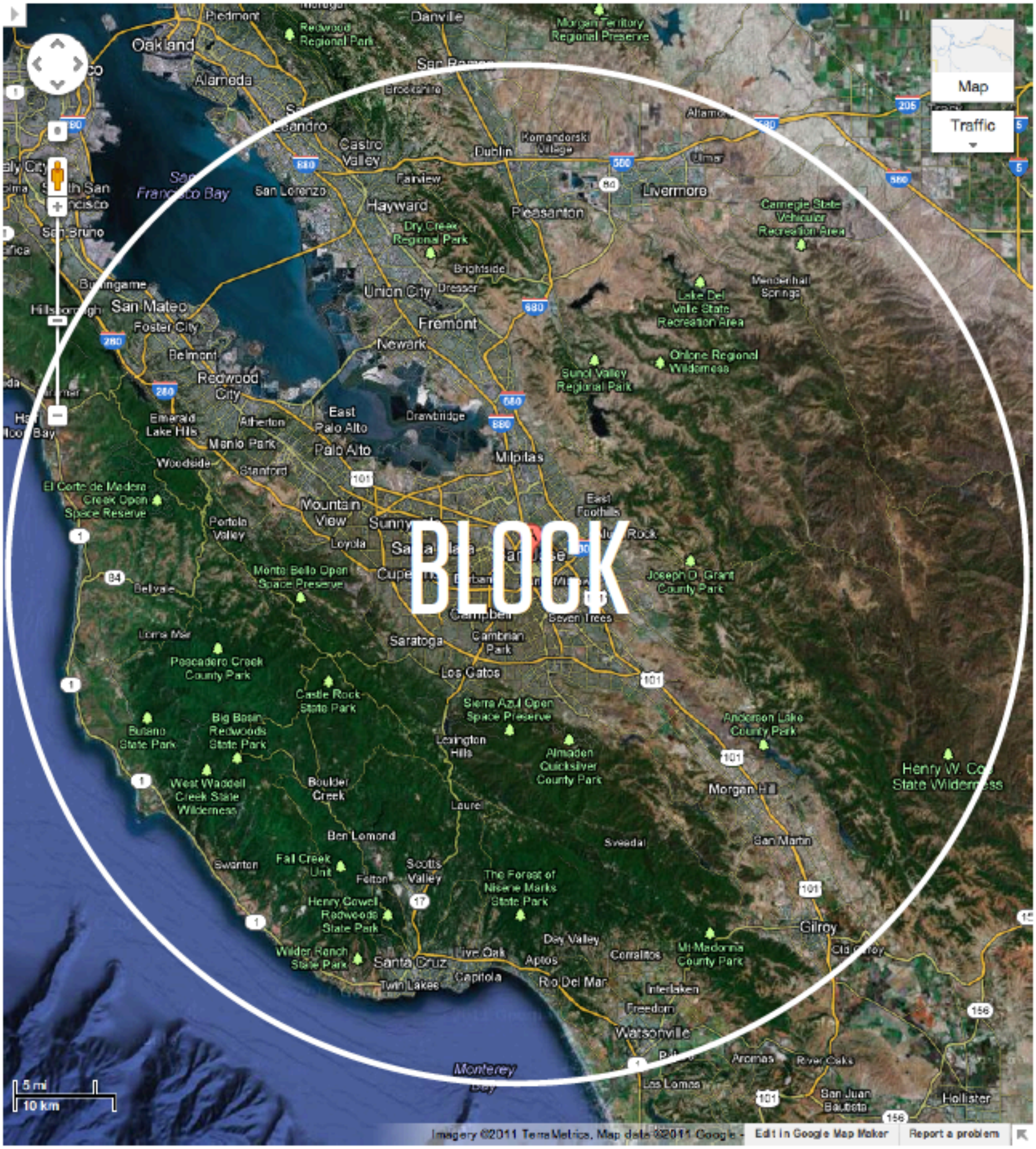
Sun Blocker



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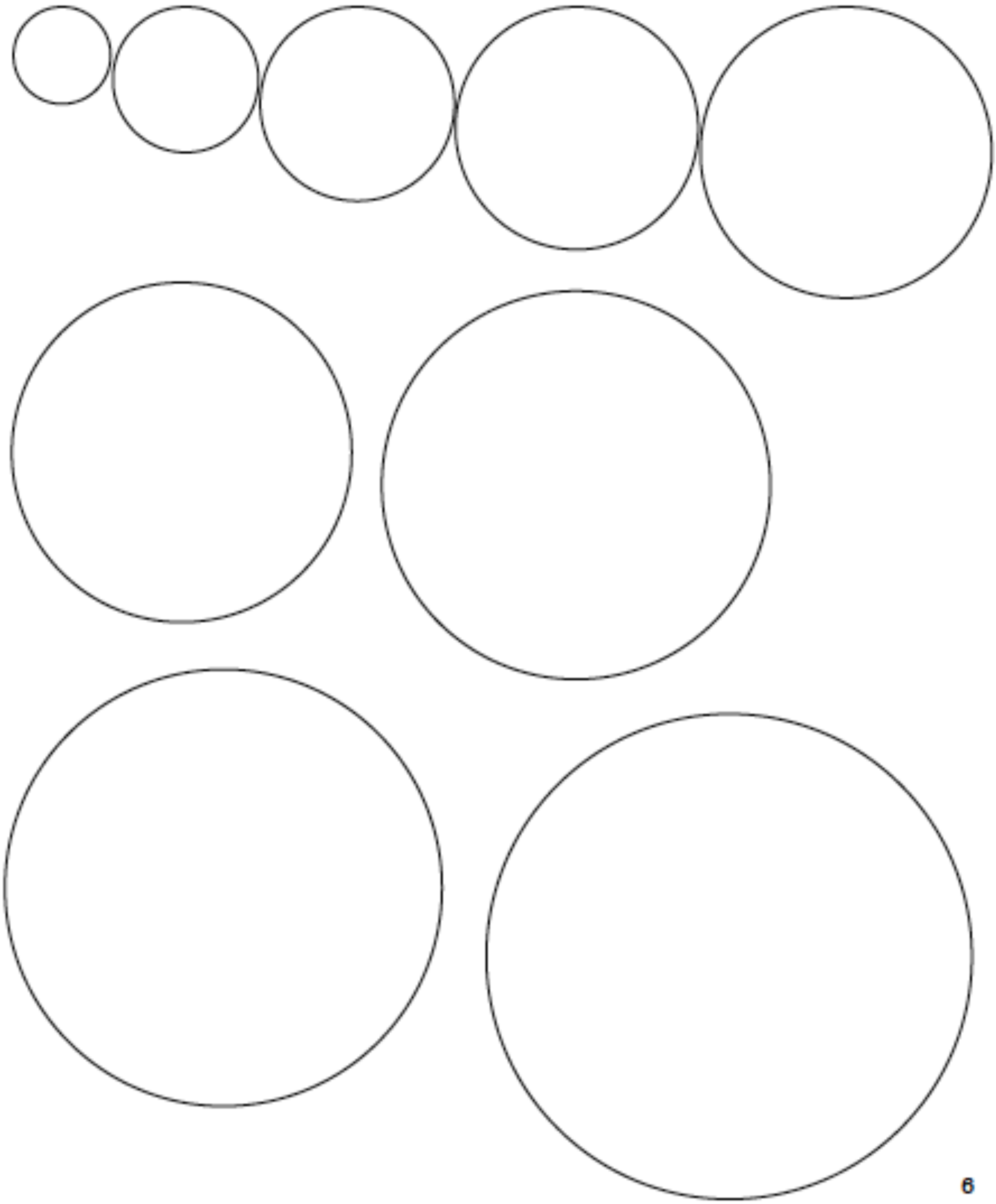

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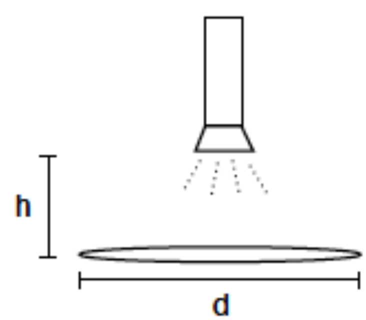
Sun Blocker

d	h
diameter of blocker	*distance from blocker to light*



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Day 2

Essential Question	Lesson Objective	Standards
How do we use math to find the height of things that are difficult to measure?	Students determine if two shapes are similar or not, given all measurements. Students explain why two shapes are similar or not, given all measurements. Students define what it means to be similar, in their own words.	G-SRT.1. Verify experimentally the properties of dilations given by a center and a scale factor. G-SRT.2. Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides. MP.3. Construct viable arguments and critique the reasoning of others.

Activities

1. Teacher Talk. Teacher will transition from the Simpsons sunblocker problem to the more abstract world of geometry. “We will study mathematical representations that can be used later to model reality. We will then return to measuring heights”.

2. Introduce “Same Shape.” The purpose of this activity is for students to begin developing an informal intuition about what “same shape” means. Students should justify their explanations.

Independent Work	Set the scene for this unit of study. Show images of things whose height might be interesting to measure.

	Ask students to generate a list of objects whose height might be interesting to measure.
Group Discussion	Students share their pictures with members of their group and share their definitions of "same shape." Groups come to internal consensus on their definitions.
Class Debrief	Groups report out their definitions of "same shape." The class come to consensus on their definition. This definition should be recorded so that it can be revisited later in the period.

3. Are These Two Shapes Similar? The purpose of this activity is to use and refine the class' current definition of similarity by first examining pairs of shapes with no angles or side measurements and then examining the same pairs with the measurements given.

Group Work	In groups, students discuss whether or not pairs of shapes are similar using the first page of the Similar Shapes task-card.
Class Debrief	Groups share out their initial impressions about which shapes are similar. The teacher records these ideas, noting conflicts. Then the teacher motivates the next part of the task to resolve these conflicts by using measurements.
Group Work	Students discuss in groups whether or not pairs of shapes are similar using the second page of the Similar Shapes task-card. Certain pairs are meant to target possible misconceptions. The teacher can circulate to check in with groups. The teacher might choose to pre-select certain groups to share out ideas during the debrief. Student thinking and misconceptions: • Classifications. Students may see two right triangles as having the same "shape" by classification even if they are not similar. (e.g. #4 on the task-card) • Proportional sides. Students may think that proportional sides are sufficient (e.g. #5). • Transformations. Students may not see shapes as similar when they have been rotated or reflected
Class Debrief	Class debrief the comparison activity and share pairs they consider similar. New ideas about the definition of "same shape" will be recorded. Teacher talking points: • Mathematical goal. The final mathematical goal of this debrief is to collaboratively construct a definition of similar shapes that contains both congruent angles and proportional sides. • Transformations and similarity. If the notion of transformations, such as rotations and reflections, has not come up already in the conversation, the teacher can use page 3 of the

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	Similar Shapes task-card to initiate a conversation about a square and a similar but rotated square (i.e. a "diamond"). Are they similar? Why or why not?

5. Exit Ticket

Independent Work	This exit ticket will ask students to explain using their own words and pictures what it means to for two shapes to be similar. This formative assessment of the concept of similarity will guide the teacher in planning for the following day.

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Similar Shapes

Put a check mark next to the pairs you think have the “same shape.” If you and your neighbor disagree, try to come to an agreement. Then, write register your thoughts in the Google Form or some other communication tool.

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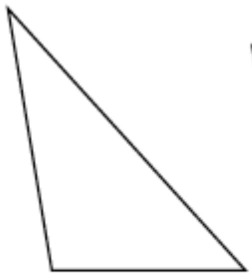

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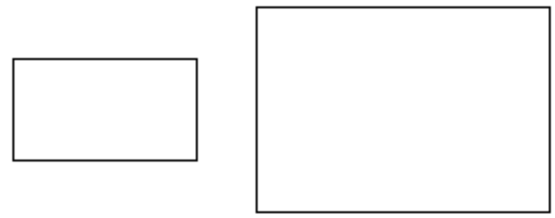

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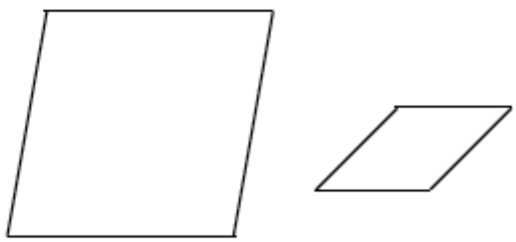

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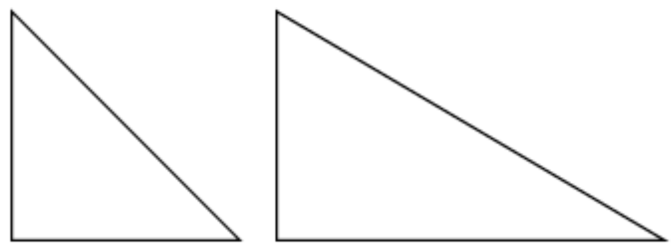

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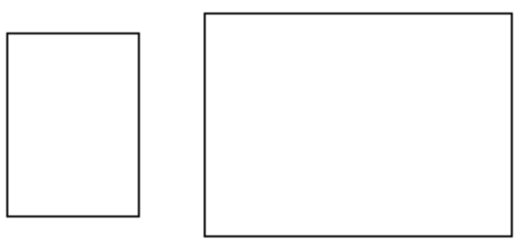

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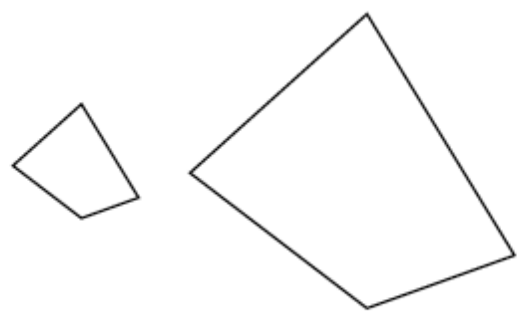

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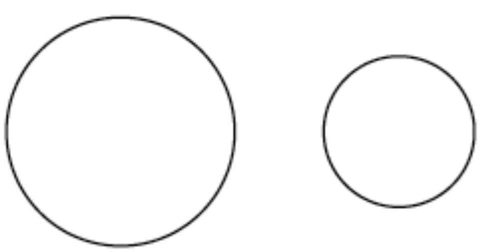

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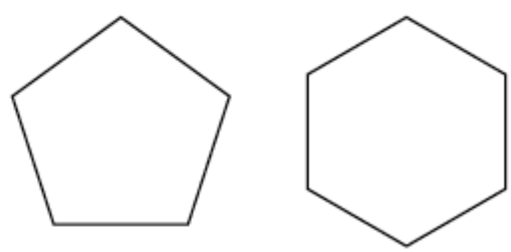

Similar Shapes

Now develop a rule that will help somebody else decide if two shapes are similar. Be very specific. Write the definition down.

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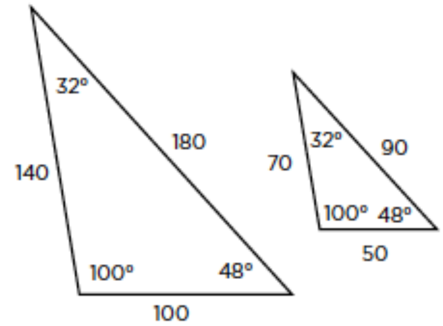

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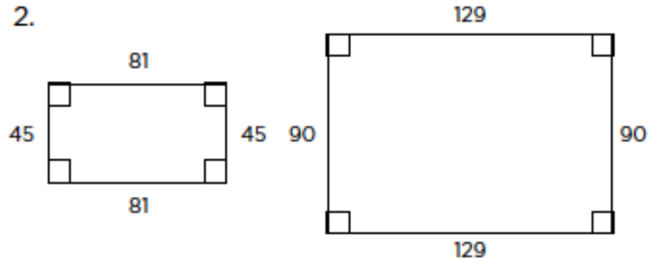

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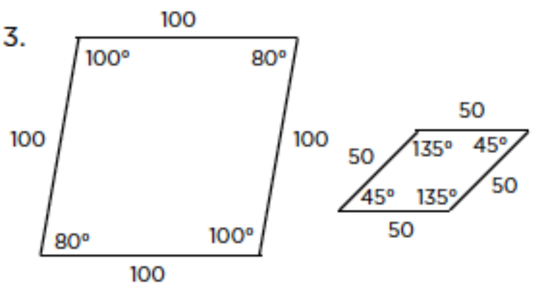

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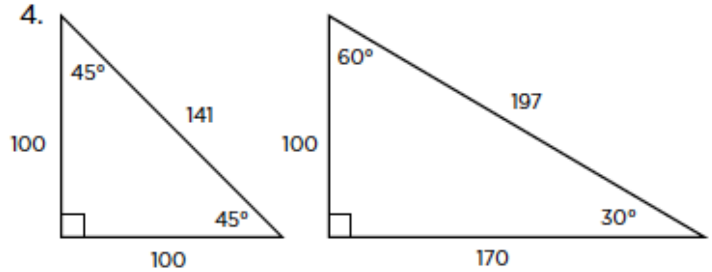

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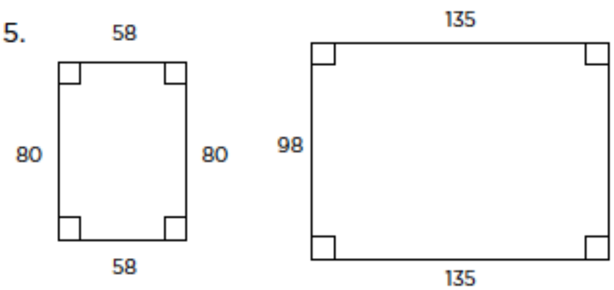

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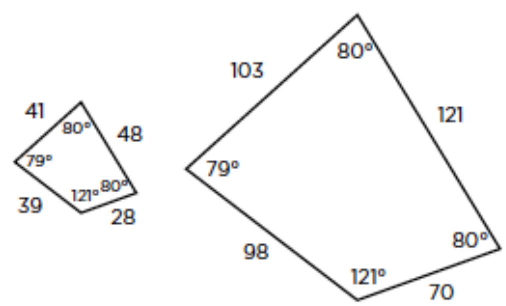

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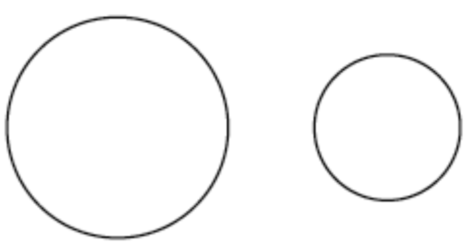

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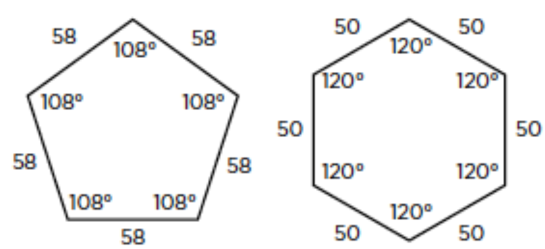

Similar Shapes

Are these similar? Why or why not?

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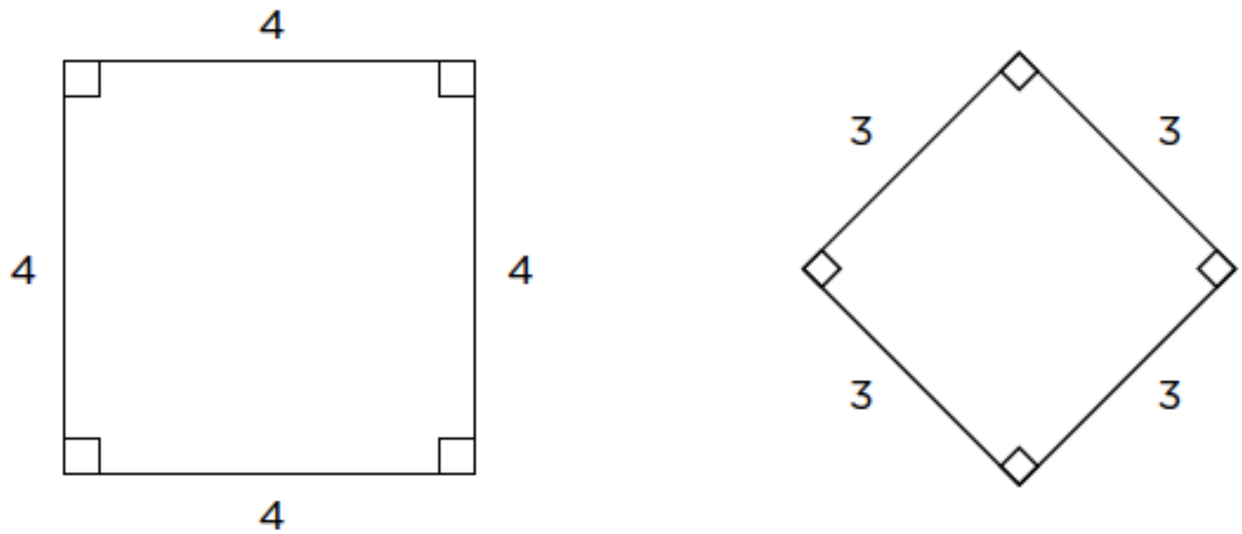
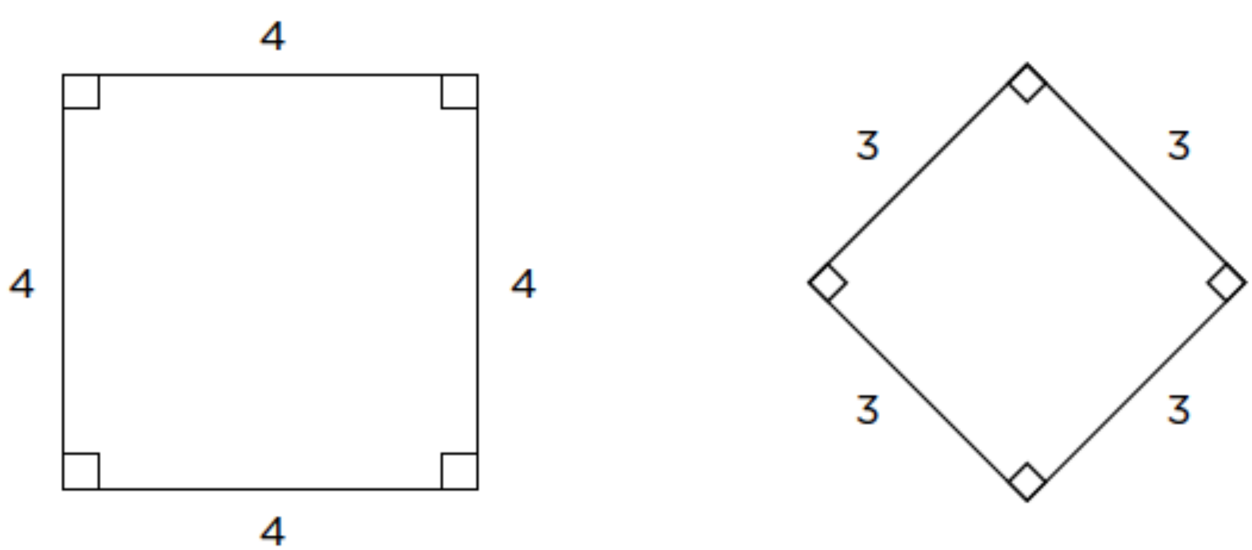


Similar Shapes

Are these similar? Why or why not?



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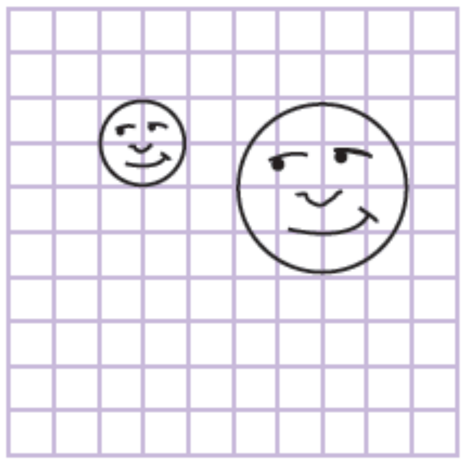
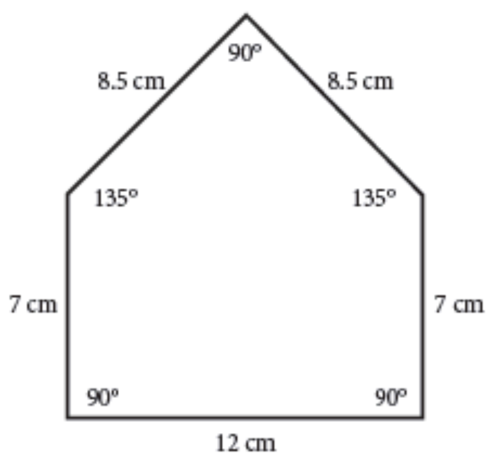
Draw the Same Shape

1. Draw a small, simple picture on a sheet of grid paper. Then draw another version of your picture that is exactly the same shape as the first one. Your second drawing should be larger than the original. You have to decide what "exactly the same shape" means to you.



2. Renata was making illustrations for a book on basic drawing techniques. She came up with this figure as the first step for drawing a house.



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Kim decided he liked this shape, but he needed a bigger version of the house for his work.

a. Give a set of length and angle measurements that you think Kim might be able to use in place of those shown in the diagram. State clearly which measurements change and which, if any, do not.

b. Carefully draw a diagram that has your suggested measurements and compare it with Renata's. Does it have the same shape?

3. Consider the following pairs of figures. In each case, state whether you consider them to be the same shape or not. Explain why.

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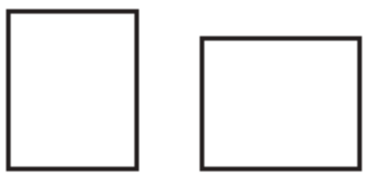
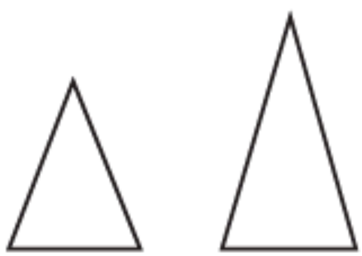


b.

c.



d.

4. Based on your experience with Questions 1 through 3, write your ideas in response to this question.

How can you create a diagram that has exactly the same shape as a given one?

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Day 3

Essential Question	Lesson Objective	Standards
How do we use math to find the height of things that are difficult to measure?	Students understand how to preserve similarity in a polygon.	G-SRT.2. Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides. G-SRT.5. Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures. MP.2. Reason abstractly and quantitatively. MP.4. Model with mathematics. MP.6. Attend to precision.

Activities

1. Preserving Similarity. The purpose of this activity is to give students the opportunity to evaluate different ways of reducing the size of a shape. This activity gives students an opportunity to construct arguments to accept or reject claims about preserving similarity.

Group Work	IMP Shadows, p. 20. In this activity, students will explore different ways to preserve similarity by changing different dimensions of a pentagon. Students will discuss why each of the given shrinking methods does or does not work.

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Class Debrief	As a class discuss which methods preserve similarity. The criteria for “same shape” from Day 2 will be edited and the class can move towards a formal definition of similarity for polygons: all corresponding sides have a constant of proportionality and all corresponding angles are the same.

2. Finding Lengths Using Similarity. The purpose of this activity is for students to apply the constant proportions between similar shapes.

Group Work	IMP Shadows, p. 21. Students estimate the length of parts of the Statue of Liberty using similarity. Students discuss questions such as: “How is this problem similar to the house shrinking problem or the Simpsons Sunblocker?”
Class Debrief	Students can be asked to share their methods, solutions, and thinking with the entire class.

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How to Shrink It?

Lola, Lily, and Lulu love Renata's house, but they find it a little too large for their liking.

They want to shrink the house to a smaller size while keeping it exactly the same shape.

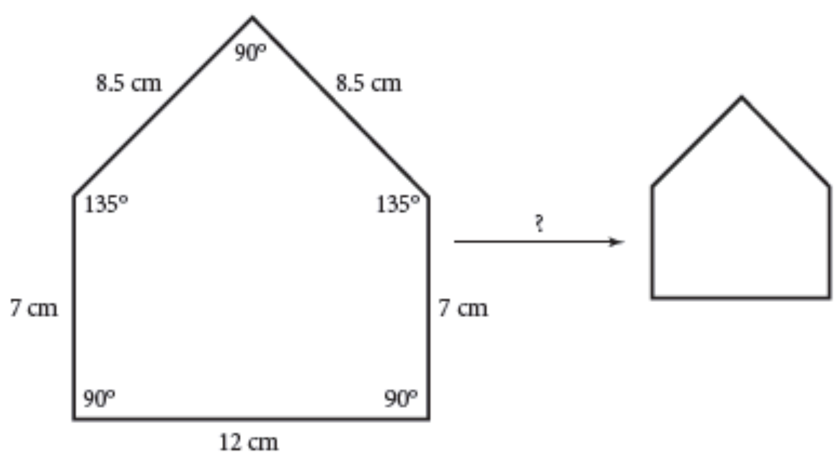
After a long discussion, each came up with a strategy for drawing the smaller house.

Lola's way: Keep all the angles as they are, and subtract 5 centimeters from the length of each side.

Lily's way: Keep all the lengths as they are, and divide all the angles by 2.

Lulu's way: Keep all the angles as they are, and divide the lengths of all the sides by 2.

Shrink the house by using each method. Show what result each method produces. Explain why each method does or does not work.



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The Statue of Liberty's Nose

The Statue of Liberty in New York City has a nose that is 4 feet 6 inches long.* What is the approximate length of one of her arms?

1. Solve the problem. You might want to think about your own nose and arms.
2. Pick two other body measurements and find the approximate length that these measurements should be on the Statue of Liberty.
3. Examine what you did with the three examples from Questions 1 and 2. How was your work the same in the three cases? How did it change from case to case?
4. State how this problem is similar to the problem of drawing a house that has the same shape as another house.
5. What connection do you see between this problem and the shadow problem?

*Measurement taken from *How They Built the Statue of Liberty* by Mary Shapiro (Random House, 1985).

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Day 4

Essential Question	Lesson Objective	Standards
How do we use math to find the height of things that are difficult to measure?	Students develop their understanding of triangle similarity. Students identify conditions that satisfy triangle similarity. (eg. AA, SSS, SAS.) Students determine whether two triangles are similar and determine the missing side or angle of similar triangles. Students construct arguments rejecting or accepting various claims to similarity.	G-SRT.3. Use the properties of similarity transformation to establish the AA criterion for two triangles to be similar. G-SRT.4. Prove theorems about triangles. G-SRT.5. Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometry figures. MP.2. Reason abstractly and quantitatively. MP.3. Construct viable arguments and critique the reasoning of others. MP.4. Model with mathematics.

Activities

1. Identifying Conditions for Triangles that Satisfy Similarity. The purpose of this activity is to provide students with the opportunity to explore conditions of triangles that preserve similarity. This activity also gives students a chance to make and test conjectures.

Launch	Teacher review: all corresponding sides are proportional and all angles are congruent to ensure similarity. Teachers can model this is Geogebra then pose: “Is there a shorter way to find out if they are similar?” The motivation should be on looking for shortcuts.
Group Work	Triangles: True or False task card (modification of IMP Shadows, p. 34). Students work in pairs using Geogebra to explore different characteristics of triangles to determine which ones will yield similar triangles. In pairs students each complete the Triangles: True or False task card. Pairs can then compare their findings with the other pair in their group. By the end of the activity, students should know that two angles (angle-angle), SAS and SSS (without emphasis on notation) is sufficient to determine similarity in triangles. Teacher Talking Points:

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	• Counterexample. Encourage students to look for the counterexample (in this case a non-similar triangle). Emphasize that a single counterexample is sufficient to disprove a claim. Student Challenges & Misconceptions: • Students may have a difficult time dealing with non-integer numbers. • Students may not see similarity as a result of reflection or rotation. • Students may aim to construct congruent triangles.
Class Debrief	Each group of four will come to consensus on each of the six conditions from the Triangles: True or False task card, and will vote their responses in a Google form. The results from the Google forms survey will be used as the focus for discussing the conditions that preserve triangle similarity, including Angle-Angle and others. The teacher should be prepared to call on various groups to share their evidence for each of the similar/nonsimilar cases.

2. Applying Similarity to Find Missing Heights.

Teacher Talk		
Group Work	Students brainstorm solution strategies. Encourage creativity and imagination!	
Class Debrief	Class Debrief	Selectively choose groups to share that have solved the problem in the following ways: 1) Proportion of height:shadow and height: shadow. 2) Proportion of height:height and shadow:shadow. 3) Proportion of shadow:height and shadow:height.* After each group shares ask, "Raise your hand if you used the same method as this group." * If one of the three methods doesn't come up, the teacher could put the method up and ask students if that works too.
Group Work	Reflection	Students individually headline what the key takeaway was for them.

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		Students should input the headline into a Google form, or similar.

Class Debrief	Selectively choose groups to share that have solved the problem in the following ways: 1) Proportion of height:shadow and height: shadow. 2) Proportion of height:height and shadow:shadow. 3) Proportion of shadow:height and shadow:height.*
Reflection	Students individually headline what the key takeaway was for them. Students should input the headline into a Google form, or similar.

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Triangles Versus Other Polygons

Triangles are special polygons. Some of them have special names. A triangle with at least two sides of equal length is called **isosceles.** If all three sides have the same length, the triangle is called **equilateral.** An equilateral triangle is a special kind of isosceles triangle.

Triangles have some characteristics that not all polygons share.

Each pair of statements includes one version about triangles and another version about polygons in general.

For each pair, first decide whether the statement is true for triangles. Then try to find a counterexample that shows the statement is not true for polygons in general.

Statements 1

If two angles in one triangle equal two angles in another triangle, then their third angles must be equal.

If two angles in one polygon equal two angles in another polygon, then their other angles must be equal.

Statements 2

If two triangles have their corresponding angles equal, then the triangles are similar.

If two polygons have their corresponding angles equal, then the polygons are similar.

Statements 3

If two triangles have their corresponding sides proportional, then the triangles are similar.

If two polygons have their corresponding sides proportional, then the polygons are similar.

Statements 4

Every triangle with two equal sides also has two equal angles.

Every polygon with two equal sides also has two equal angles.

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Group Activity

Are Angles Enough?

You've seen that the lengths of the sides of a triangle determine the triangle. Here is another way to say this.

> If the lengths of the three sides of one triangle are the same as the lengths of the three sides of a second triangle, then the two triangles are **congruent**.

What about angles? Do the angles of a triangle determine the triangle? What happens if two triangles have the same three angles?

1. Start with angles of 40°, 60°, and 80°. Each group member should draw a triangle using these three angles.
   Did the triangles all come out congruent? Were they all similar? Why or why not?
2. Do the same thing as in Question 1, but start with a different set of angles. You might draw an arbitrary triangle and then have each group member use the same angles as that triangle.
   Did the triangles all come out congruent? Were they all similar? Why or why not?
3. Go back to the angles of 40°, 60°, and 80° from Question 1. This time, decide as a group on the length each group member will use for the side connecting the angles of 40° and 60°. As before, have each group member draw a triangle using angles 40°, 60°, and 80° but also using the given length in the given position.
   Did the triangles all come out congruent? Were they all similar? Why or why not?

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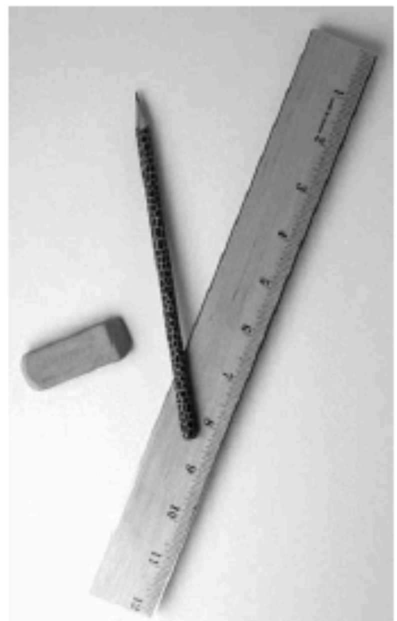
What's Possible?

You have found that the three angles of a triangle are subject to a very strict condition—their sum must be exactly 180°. This activity poses a related question about the lengths of the sides of a triangle.

Can any three numbers be the lengths of the sides of a triangle?

You may want to work on this question using physical materials, such as straws, paper strips, or pieces of spaghetti.



Experiment to find the answer to the question. You might start with the numbers 2, 3, and 4. Can you draw a triangle that has sides with these lengths? Be sure to choose a unit of length.

Try three other values, such as 3, 6, and 11. Keep making up numbers and testing them. Keep track of which sets of lengths are possible and which are not. What conclusions can you reach about the three sides of a triangle?

What About Other Polygons?

What principles can you formulate about the lengths of the sides of a quadrilateral? Play around with examples, using physical materials if you choose.

Continue with polygons that have more sides. Can you state a general principle for whether a given set of numbers can be used as sides of a polygon?

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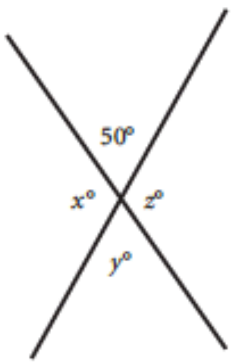
Activity

Angles, Angles, Angles

Find the designated angles in each diagram. Lines marked with arrows are parallel.

1.



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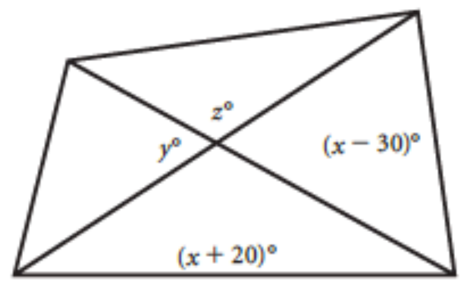
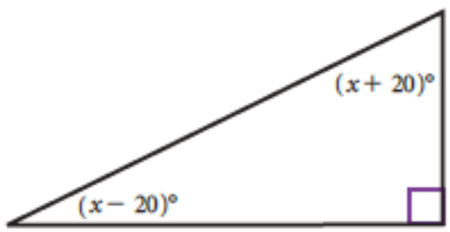


3.



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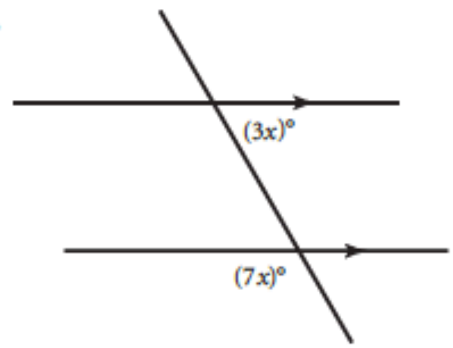

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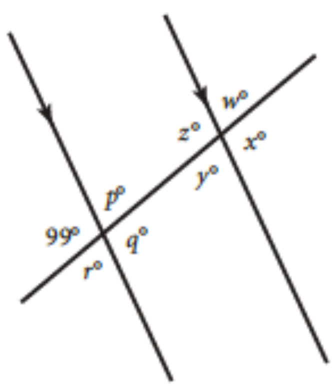
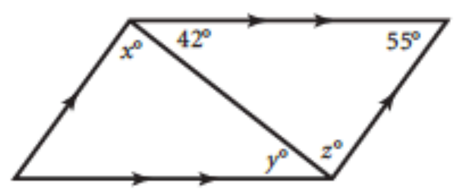


6.



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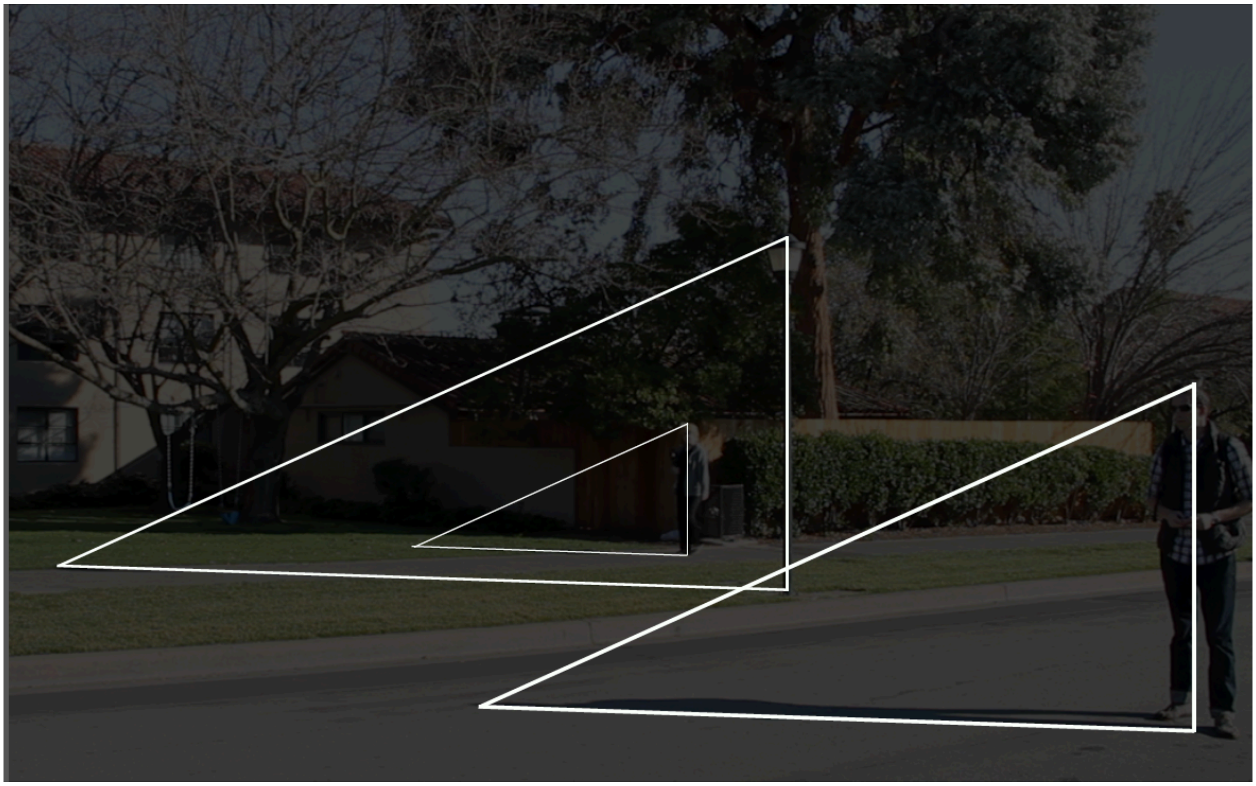

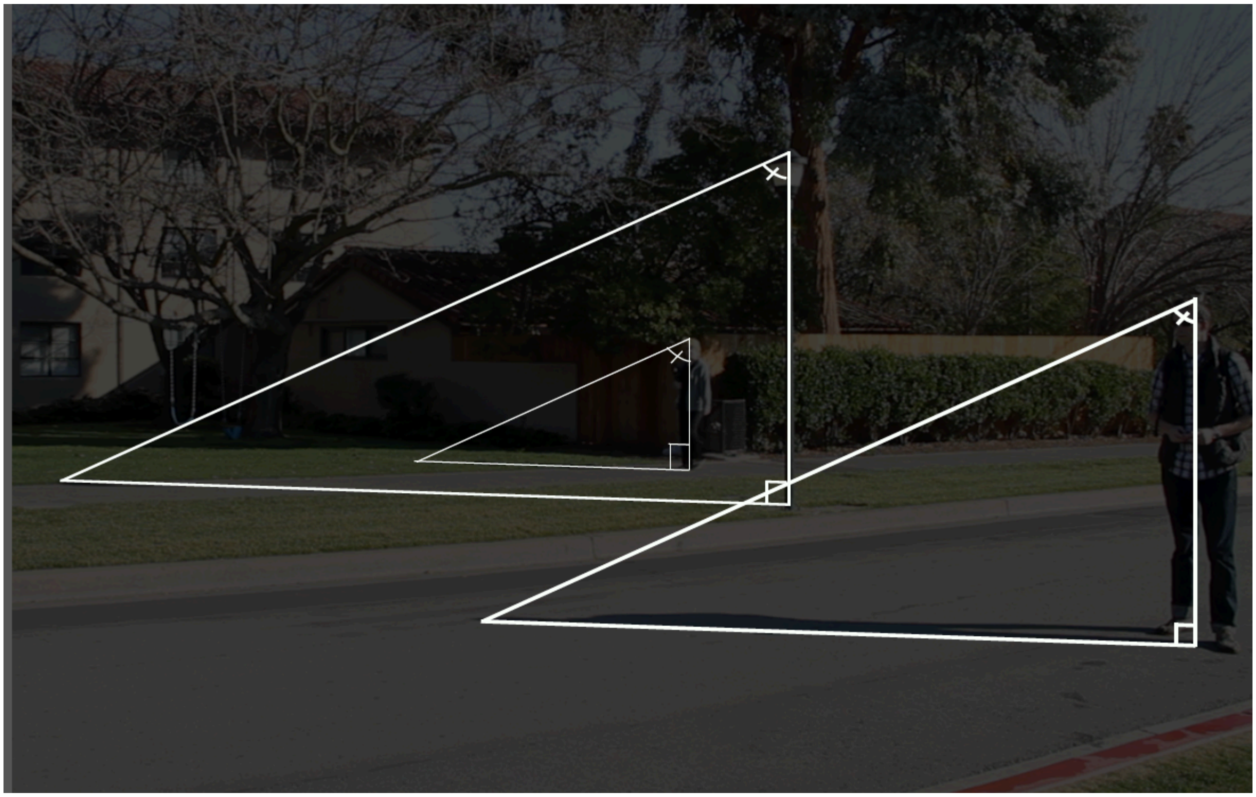

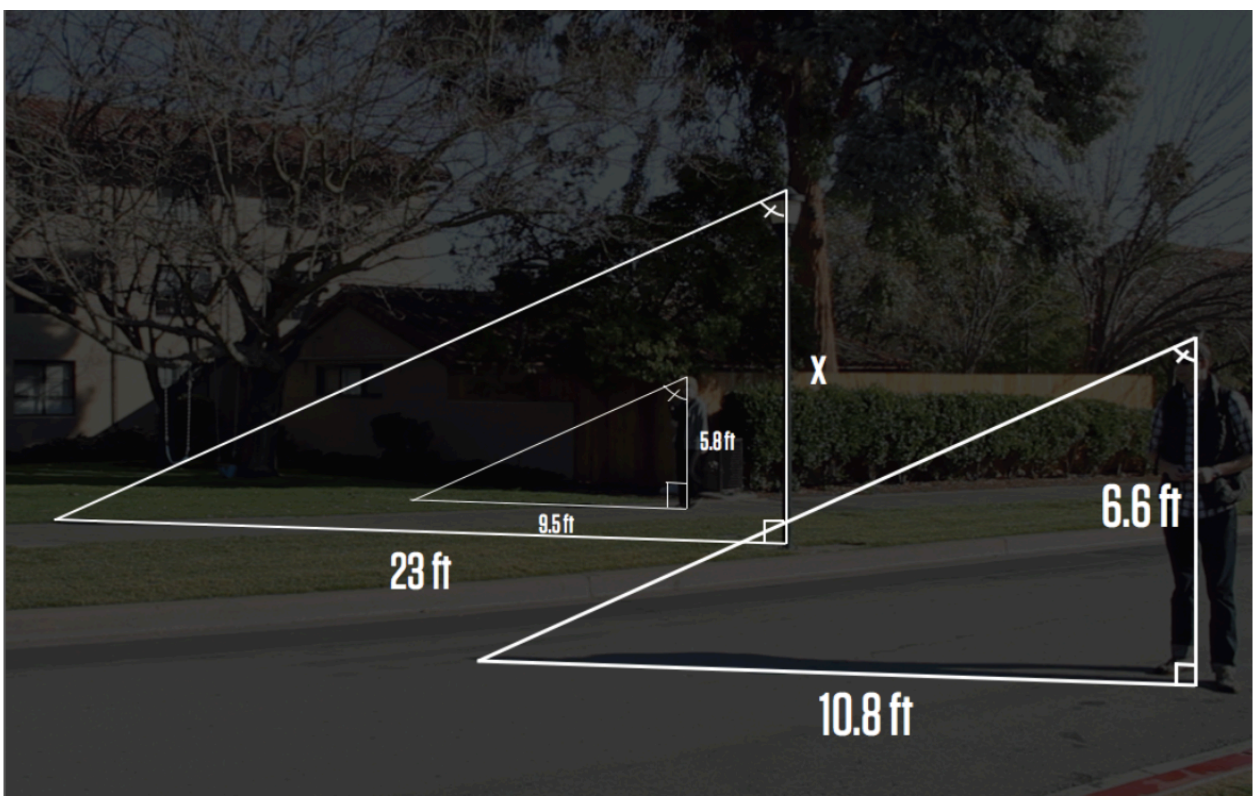

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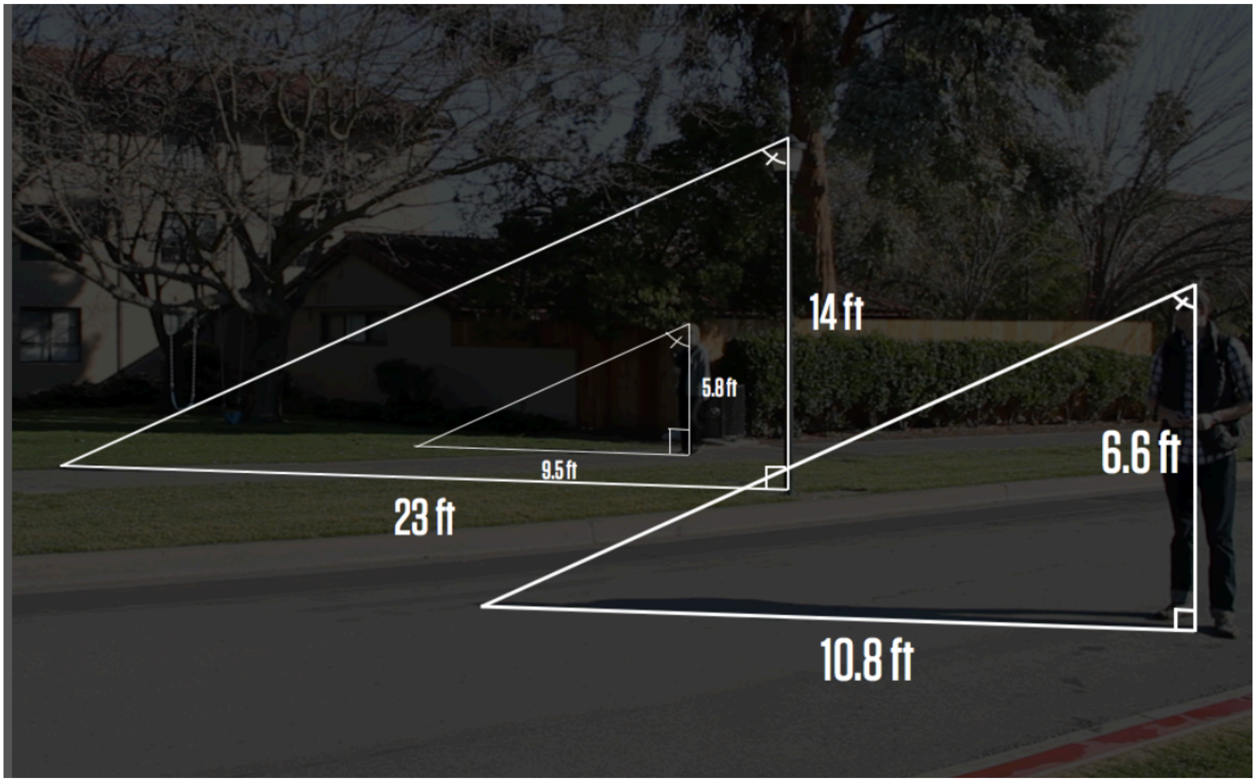
youcubed at Stanford University



Triangles: True or False

Guess if each statement below is true or false.
Label each of them with a T or an F.
Then create each scenario in Geogebra.
Sketch your creation and decide whether or not to change your guess.
If two angles in one triangle are each congruent to two angles in another triangle, then their third angles must be congruent.

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Guess	*Evidence (sketches, calculations)*	*Answer*

2. If exactly one angle in one triangle is congruent to one angle in another triangle, then the two triangles are similar.

Guess	*Evidence (sketches, calculations)*	*Answer*

3. If all the corresponding angles are congruent in two triangles, then the triangles are similar.

Guess	*Evidence (sketches, calculations)*	*Answer*

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Triangles: True or False

Guess if each statement below is true or false.
Label each of them with a T or an F.
Then create each scenario in Geogebra.
Sketch your creation and decide whether or not to change your guess.
Every triangle that has two congruent sides always has two congruent angles.

Guess	*Evidence (sketches, calculations)*	*Answer*

5. If two sides in a triangle are proportional to two corresponding sides in another triangle, then the triangles are similar.

Guess	*Evidence (sketches, calculations)*	*Answer*

6. If two triangles have all their corresponding sides proportional, then the triangles are similar.

Guess	*Evidence (sketches, calculations)*	*Answer*

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Day 5

Essential Question	Lesson Objective	Standards
How do we use math to find the height of things that are difficult to measure?	Students can revisit the Simpsons Sunblocker project and model the problem using triangle similarity. Students can prove triangle similarity for special cases of a triangle within a triangle. Students learn to identify and construct similar inside triangles. Students demonstrate understanding of inside similarity by solving problems involving a triangle within a triangle.	G-SRT.4. Prove theorems about triangles. G-SRT.5. Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometry figures. MP.2. Reason abstractly and quantitatively. MP.3. Construct viable arguments and critique the reasoning of others. MP.4. Model with mathematics.

Activities

1. Warm Up: Revisiting the Simpsons.

Group Work	Remind students of the Simpson's Sunblocker problem by projecting an image of the video clip. Students are then asked to individually model the situation by drawing a picture: "In your notebooks sketch a diagram that represents the light, disk, and the shadow from the sunblocker problem." Once sufficient time has past, students will share their diagrams with their group mates and respond to the question, "How many triangles are in this diagram and how would you know if they are similar?
Class Debrief	Preselect a student or group to draw their diagram on the board. Teacher can ask the class, "What triangles do you see in this picture?" Teacher can then ask, "Are any of these triangles similar? How do you know?" If there are other interesting diagrams, the teacher may want to have more than one group share. At some point during the debrief requirements for triangle similarity should be mentioned, including: proportionality of sides or angle congruence (review of Day 4). Student Conceptions: • Nested Isosceles Triangles --The diagram does not contain a vertical line. • Nested Right Triangles -- The diagram shows the above, but with a vertical altitude.

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	• Concentric Circles -- This diagram represents an eagle eye view of the situation, but does not afford the use of triangles. • 3-Dimensional Drawing -- The diagram may be a cone. Teacher should encourage students to see the triangles in the picture.

2. Exploring Triangles Within Triangles. The purpose of this activity is to give students a chance to make and test conjectures about triangles. Students will communicate their mathematical ideas to their peers.

Group Work	Give students the "Inside Similarity" worksheet (a modification of IMP Shadows, p. 49). Every student should receive his/her own worksheet. Students in groups will find as many ways to make an inside triangle that is similar to the outside one. Students will determine conditions that make the inside triangle similar to the outside triangle. Students should come to consensus as to why their conditions do or do not work. Teacher Talking Points: • Properties of Similar Triangles. Teacher encourage students to work deductively from properties. For example, "If I know I need two congruent angles, how could I get two congruent angles?" • Measurement. Teacher will encourage students to use tools such as protractors and rulers to support their investigation. • Angle Theorems. Teacher will encourage students to draw on their knowledge of angle theorems. Student Challenges & Misconceptions: • Organization. Students might not experiment systematically. Teacher might encourage students to pick two sides and cut segments across those two sides first. • Identifying Corresponding Parts. Students may reason proportionally about segments that are not corresponding parts of triangles.
Formative Assessment	When each group reaches the checkpoint on "Inside Similarity", the teacher should come over with a mystery triangle. The group can be asked to construct an inside triangle that is similar to the big one. The teacher can randomly select one student from the group to explain how they know the inside triangle is similar to the outside triangle.
Class Debrief	Groups share out how they know if an inside triangle is similar to an outside one. In this conversation teacher should highlight the fact that if one side of the inside triangle is parallel to a side of the outside triangle then the two triangles will be similar. Students should justify their answers. Student Conceptions:

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	• Shapes "look the same." A response like this one would require more teaching prompting, asking for clarification of what "same" means (see day 3). • Proportions of the cut sides (not corresponding parts) are equal. Note: Although this observation is true, it is not sufficient for triangle similarity. Teacher should push towards corresponding parts. • Proportions of the corresponding sides are equal. • Angles of both triangles are the same either through measurement or parallel line/angle theorems. • One side of the inside triangle is parallel to one side of the larger triangle. (Teacher follow up: Why does this mean that they are similar?)

3. Applying Triangles Within Triangles. The purpose of this activity is to give students a chance to apply their knowledge of inside triangle similarity.

Group Work	Using their discoveries from the previous activity regarding inside similarity, students will 1) Determine the placement of a specific sunblocker given the size of San Jose and the distance to the sun (assuming noon). 2) Generate the size and placement of an additional sunblocker. Students can pick any size and any distance that is reasonable and works.

4. Extension.

Extension Activity	IMP Shadows, p. 57. Students will solve an inside triangle problem involving light and shadows. In this assignment, students must be able to identify the pair of similar triangles in a shadow image and solve for different unknown lengths.

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Exploring Inside Similarity

How do you make small triangles inside larger ones so that the small ones are similar to the large ones? The diagram shows three copies of the same triangle. In the bottom two triangles, a dotted line segment has been drawn that connects two sides of the triangle and cuts off a smaller (shaded) triangle.

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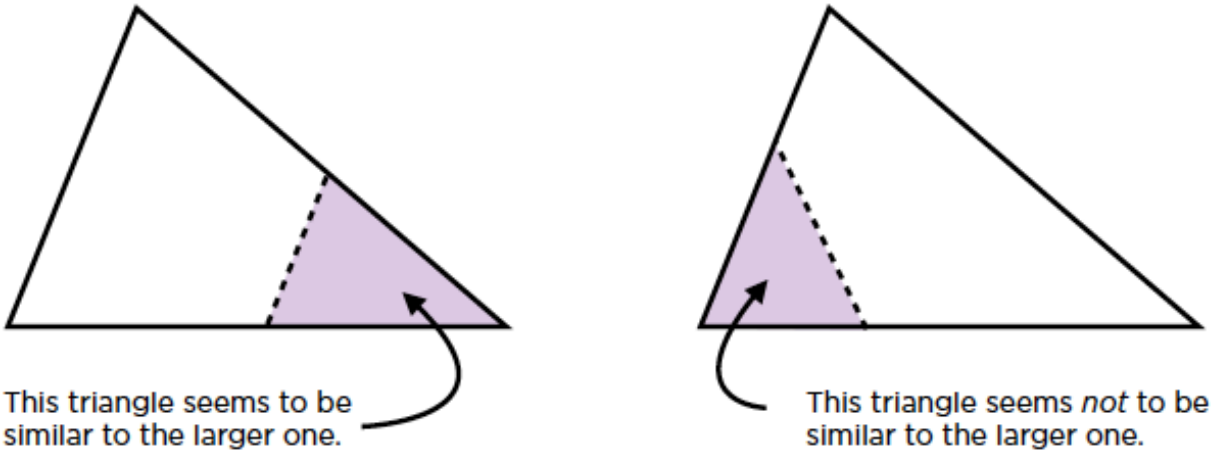
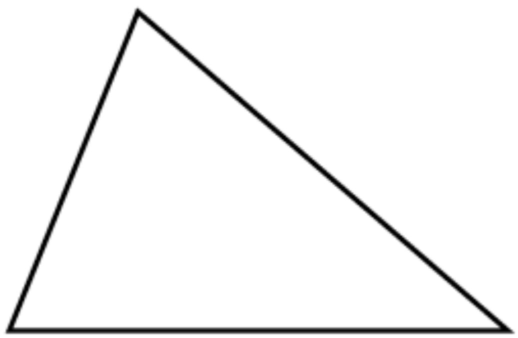


1. Find as many ways as you can to draw lines that cut off small triangles similar to the original triangle.



2. Describe using words and pictures how to construct an inside triangle that is similar to the larger triangle.

3. How do you know that your method in (2) produces similar triangles?

Checkpoint

Construct a similar triangle inside a mystery triangle. Be prepared to explain your thinking.

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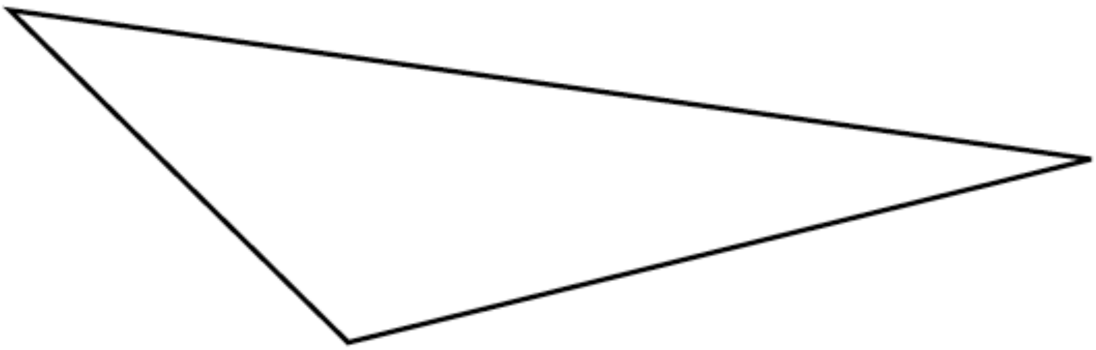
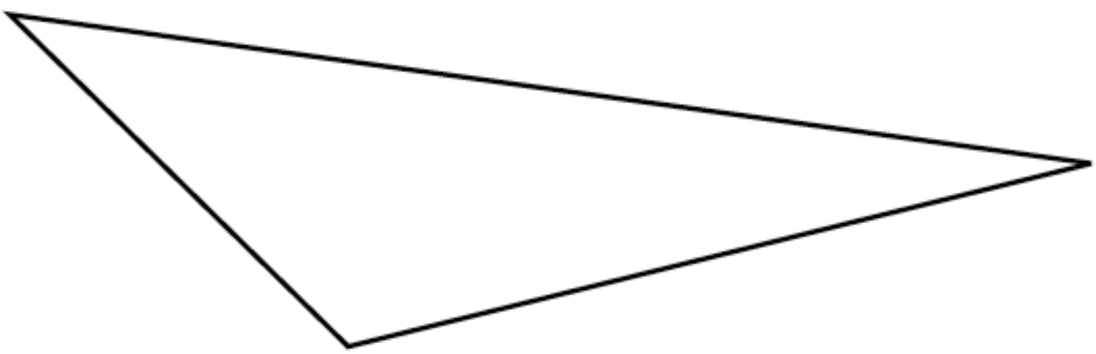


Checkpoint

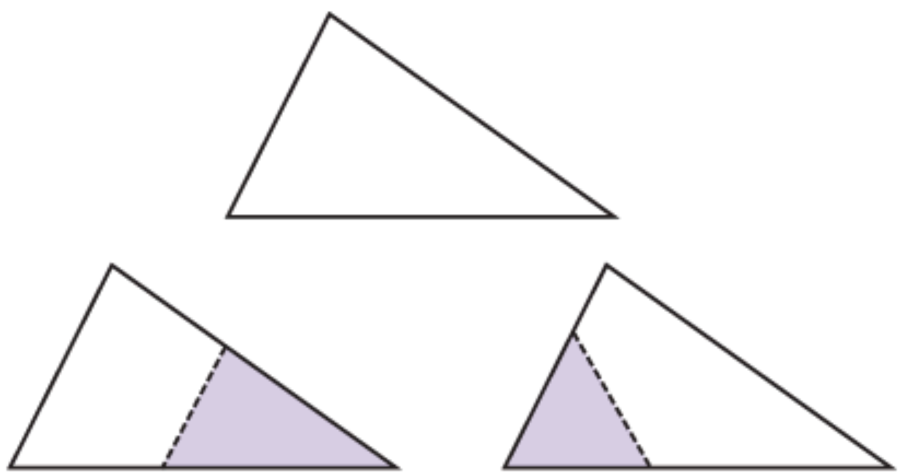
Construct a similar triangle inside a mystery triangle. Be prepared to explain your thinking.



Inside Similarity

How do you make small triangles inside larger ones so that the small ones are similar to the large ones?

The diagram shows three congruent triangles. In the bottom two triangles, a dotted line segment has been drawn that connects two sides of the triangle and cuts off a smaller (shaded) triangle.



In the first case, the smaller triangle appears to be similar to the larger one. In the second case, the smaller triangle seems not to be similar to the larger one.

Your task is to investigate and report on the difference between these two cases. If you connect points on two sides of a triangle, when does the smaller triangle created in this way come out similar to the original?

You may want to begin your investigation by tracing the original triangle. Then experiment by drawing some segments on your tracing. Find as many ways as you can to draw lines that cut off small triangles similar to the original triangle.

Describe in words those segments that can be used to cut off a small triangle that is similar to the larger one, and explain your answer.

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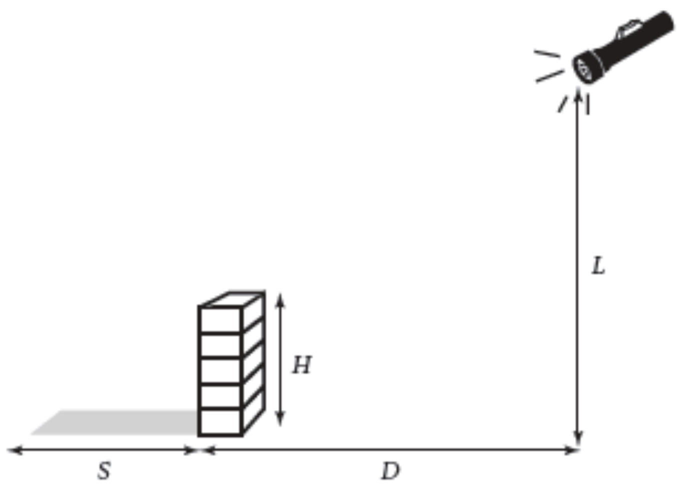
A Shadow of a Doubt

By now, you have probably developed a diagram similar to this one to represent the lamp shadow problem.

In terms of this diagram, the length of a shadow (S) depends on three things: the height of the light source (L), the height of the object casting the shadow (H), and the distance along the ground from that object to the light source (D).



1. What triangles do you see in the diagram? Which of them are similar? Why must these triangles be similar?
2. Use your knowledge of similar triangles to write an equation that expresses a relationship among the four variables.
3. Here is some information about a specific situation involving Shoshana, a lamppost, and her shadow. Shoshana is standing 9 feet from the lamppost. The light at the top of the lamppost is 12 feet high, and Shoshana is 5 feet tall. Use your equation from Question 2 to find the length of Shoshana's shadow.
4. Find the length of the shadow if $L = 11$, $H = 6$, and $D = 13$.
5. Write a description in words of how to find the length of a shadow when L, H, and D are given.

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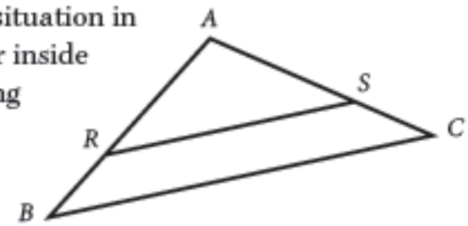
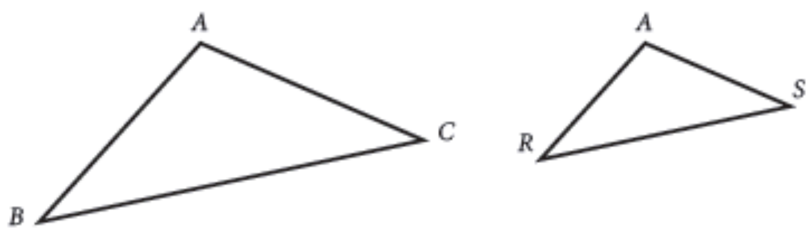
More Similar Triangles

Your work with triangles will often involve a situation in which the similar triangles are overlapping, or inside one another. Here is an example of overlapping similar triangles. Do you see the two similar triangles?

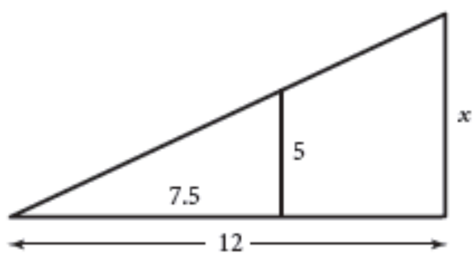


If you were to separate them and redraw them, the two triangles would look like this.

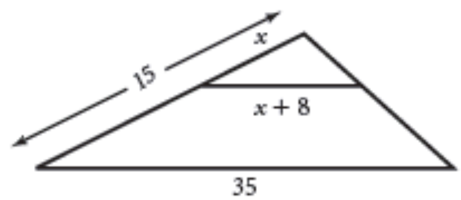


Find the unknown lengths in these pairs of overlapping similar triangles.

1.



2.



Use your calculator to solve these proportions for the unknown side length.

3. $\frac{4.6}{27.6} = \frac{2.5}{x}$

4. $\frac{301}{d} = \frac{426}{5964}$

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Day 6

Essential Question	Lesson Objective	Standards
How do we use math to find the height of things that are difficult to measure?	Students to explore angle relationships and discover that the angle of approach is equal to the angle of departure. Students use the angle of approach and the angle of departure to establish similar triangles. Students demonstrate their understanding of similarity by measuring the height of objects.	G-SRT.5. Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometry figures. MP.2. Reason abstractly and quantitatively. MP.3. Construct viable arguments and critique the reasoning of others. MP.4. Model with mathematics.

Activities

1. Mirror Lab

Launch	Ask for a student volunteer, then place a mirror on the floor and sit in a chair. The teacher asks the other students to guess where the volunteer should stand so that he/she can make eye contact with the teacher through the mirror. The volunteer moves accordingly. After two or three guesses, the volunteer moves to make eye contact with the teacher through the mirror. Teacher can then say, "Today we are going to explore the use of mirrors. By the end of today, you will be able to figure out exactly where the student should stand to make eye contact."
Group Work	Materials: Mirror, string, protractor, ruler, tape. In this activity students will be asked to model various scenarios of making eye contact using a mirror. Students will be asked to draw a diagram of each scenario, and measure and label all possible measurements in the diagram. From these measurements students will begin to make observations about the scenarios. Ultimately, students should be able to predict, with relative accuracy, where an individual should stand to make eye contact with another individual. Teacher Talking Points: • Measurement. Teacher can encourage students to use tools such as protractors and rulers to support their investigation.

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	Teacher can also assist students in selecting a precise unit of measure (e.g.; inches, cm). • Observations. Teacher can assist students to make observations about relationships within the scenarios. Teacher can encourage students to keep a record of their observations. • Similarity. When students observe the proportionality of sides, the teacher should push towards for an explanation of why. This discussion should push students to observe angles and identify similarity. • Connection to Billiards. Teacher should ask students how this activity is connected to the billiards activity.
Group Work	Student Challenges & Misconceptions: • Organization. Students might not experiment systematically. Teacher might encourage students to replicate the launch scenario (one person sitting, the other standing) • Finding all measurements. Students may not realize the need to measure the angles. Teacher can encourage students to use two strings taped to the mirror to measure angle of approach and angle of departure. Students may not realize that
Formative Assessment	Once groups have completed the lab, the teacher can give the group a mirror scenario and require the group to identify where the triangle should be placed. The groups could be given one or more chances to identify where the student should stand. The teacher could randomly select one student from the group to explain their solution.

2. Using Mirrors to Measure Height.

Teacher Talk	Teacher launches the activity. "On day four, we measured the height of objects using shadows. However, if it's a cloudy day, we can no longer use shadows. Teacher shows an image of an individual standing next to a tall object on campus, without any shadows. A mirror might be able to help.
Group Work	Students brainstorm solution strategies. The teacher can prompt them to look for similar triangles in the images. Students will calculate the height of the tall object.
Class Debrief	Groups can share out and justify their solutions and strategies.

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Experimenting with Mirrors

In this activity you will explore where to place a mirror so that you can make eye contact through the mirror:

1. Guess where one should stand to make eye contact.
2. Draw a diagram of the situation.
3. Measure and record all relevant information.
4. Record any observations.

1. Scenario 1:______________________________	2. Scenario 2: ______________________________
3. Scenario 3: ______________________________	4. Scenario 4: ______________________________

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Day 7

Essential Question	Lesson Objective	Standards
How do we use math to find the height of things that are difficult to measure?	Students demonstrate their understanding of similarity by measuring the height of objects. Students explain their mathematical processes for solving problems with similarity.	G-SRT.5. Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometry figures. MP.2. Reason abstractly and quantitatively. MP.4. Model with mathematics.

Activities

1.Launch

Teacher Talk	Project solution strategies for measuring heights of tall objects using a mirror. (See Day 6.) Teacher explains the instructions for the Mirror Lab and models the measuring tools. Teacher talking points: • Precision and rounding: Consider briefly discussing issues of precision, rounding, and possible sources of error. • Units: If relevant, it could be helpful to review different units that could be used in this lab based on the materials.
Group Work	Groups can go outside and collect data to calculate the heights of two different objects using a mirror. They will first guess the height. See the Mirror Lab document.
Group Work	Once the groups have collected enough data, they can create posters outlining their strategies, procedures, and solutions. Mention that the goal of the posters is to communicate their mathematical methods as clearly and convincingly as possible. The posters should stand alone, since they will not be near their posters during the gallery walk. Remind students that other student will be closely examining their mathematical work on the posters to learn about each others ideas.

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	Teacher talking points: • Convincing communication: Ask students to think about what would make a poster convincing. Might consider mentioning clear organization and structure, multiple representations, detailed explanations, labeled diagrams, use of color, etc. • Similarity: Prompt students to be explicit about their use of similarity by using words and diagrams to show their work.

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Mirror Lab

In this activity you will measure the height of three tall objects using a mirror.
1. Guess the height of the object.
2. Draw a diagram of the situation.
3. Measure and record all relevant information.
4. Label the diagram appropriately.
5. Calculate the height of the object.

1. Object: ____________________________	2. Object: ____________________________
3. Object: ____________________________	4. Object: ____________________________

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Bouncing Light

If you take a flashlight and shine it into a mirror, the light will "bounce off" the mirror. In this activity, you will look at the angles involved in such bouncing light.

Set up your flashlight and mirror in a way similar to that shown in the sketch. One person can hold the mirror, another can shine the flashlight at the mirror, and a third can mark the path along which the reflection of the light leaves the mirror.

The angle between the mirror and the incoming ray of light from the flashlight is called the *angle of approach*. The angle between the mirror and the ray bouncing off the mirror is called the *angle of departure*.

1. Use a protractor to measure the angle of approach and the angle of departure. You may want to trace the path of the flashlight's beam to and from the mirror onto chart paper.

   What do you notice about the relationship between the two angles?

2. Repeat this experiment, but change the angle at which the beam goes into the mirror. Do this two more times, each time with different angles. Does the relationship that you observed between the angles always seem to hold true? Write down what you've noticed about the relationship.

3. Hold the mirror in different positions to look at other things in the classroom. What do you notice about the position of the mirror? Write about the relationship between your observations in Question 2 and how you have to hold the mirror. Try to explain what looking at something in a mirror has to do with bouncing light.

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Day 8

Essential Question	Lesson Objective	Standards
How do we use math to find the height of things that are difficult to measure?	Students will examine and critique the work of their peers. Students will review concepts related to similarity. Students will explain mathematical concepts using multiple representations.	G-SRT.3. Use the properties of similarity transformations to establish the AA criterion for two triangles to be similar. G-SRT.4. Prove theorems about triangles. G-SRT.5. Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometry figures.

Activities

1. Gallery Walk.

Group Work	Pairs or groups of students will walk around the room looking at other groups' posters from Day 7. Students will leave comments on posters using post-it notes. They will draw on a list of prompts, which is in unit1_8_gallerywalk. These are both method and communication prompts. Teacher talking points: • Norms – Consider highlighting the importance of giving helpful feedback. Set rules for what are not acceptable. • Convincing - Begin to address what it means to be convincing as a preview of the justification mini- unit.

2. Newsletter.

Class Debrief	The class can discuss its observations from the gallery walk. Some possible prompts for the debrief: • How many different methods did you notice on the posters? • What did you learn about similarity from looking at other group's posters? • What helped make a poster convincing for you? Why? • If you were to make another poster, what would you do differently to make your poster more convincing?

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Individual Work	Students can create a newsletter of what they know about similarity so far. See the Similarity Newsletter document.

I noticed that you did ____________ to solve the problem while my group did ____________.

Why did you choose to do ____________ to solve the problem?

I noticed that you did ____________ to solve the problem, which was the same as my group.

I learned ____________ from looking at your poster, which was new for me.

Communication Prompts

The focus of these prompts is giving feedback on the quality of the *communication* of the posters, rather than the methods used.

I didn't understand ____________ on your poster because ____________.

I really like the way you did ____________ because it helped me to ____________.

I wish you had included ____________ because it would have helped me ____________.

I agree with your choice to include ____________ because it ____________.

Something that might have made your poster more clear is ____________ because ____________.

The most convincing part of your poster is ____________ because it helped ____________.

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Similarity Newsletter

You are writing a newsletter to share your learning in this similarity unit with your family and friends. You'll have the chance to describe your understanding of similarity and write about why this mathematical idea is important. You'll also describe a couple of activities that you worked on that were interesting to you.

In creating your newsletter, you can draw on the following resources:

- Photos of different activities.
- Sketches.
- Cartoons.
- Interviews/surveys.

Activities

To refresh your memory, here are some of the activities we've worked on in this unit:

- Simpsons Sunblocker
- Drawing the Same Shape
- Preserving Similarity: Keeping a pentagon the same.
- Statue of Liberty: Finding the size of her nose.
- Triangle —"True or False: Using Geogebra to check similarity conditions.
- Shadow Lab: Using shadows to measure height.
- Exploring Inside Triangles: How to make a similar inside triangle.
- Exploring Billiards
- Mirror Lab: Exploring where to stand to make eye contact.
- Using Mirrors to Measure Height
- Khan Practice
- Gallery Walk
- Poster Presentations

Sections

Include the following sections in any order.

Similarity

Explain what similarity means in two ways.

- Give an intuitive description of similarity.
- Give a formal definition of similarity.

Make sure to express your ideas in different ways. Consider using words, diagrams, pictures, numbers, and equations.

Similarity Activity Reflections

Choose two activities in the unit that helped you understand and use the concept of similarity.

For each activity:

- Explain why you chose each activity.
- Explain what you learned about similarity through the activity.

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• Explain what was challenging about the activity.
• Explain the strategies you used to move forward.

Other Activity Reflections

Choose one additional activity that helped you learn a mathematical idea or process.

• Explain why you chose each activity.
• Explain the big mathematical idea you learned from the activity.
• Explain what was challenging about the activity.
• Explain the strategies you used to move forward.

Closing

Write a summary for the newsletter that addresses the following:

• What is similarity useful for?
• What questions do you still have about similarity?

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Similarity Newsletter

Rubric

	Something	Something	Something
Something else.	TK	TK	Tk
Something else.	TK	TK	Tk
Something else.	TK	TK	Tk

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Day 9

Essential Question	Lesson Objective	Standards
How do we use math to find the height of things that are difficult to measure?	Students will understand the constant relationship between sides of right triangles. They won't yet use formal terminology like "sine," "cosine," and "tangent." Students will know and be able to use the terminology for sides of a right triangle (eg."adjacent," "opposite," and"hypotenuse.") Students will be able to use trigonometric ratios for certain angles to solve for the missing sides in right triangles.	G-SRT.6. Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles. G-SRT.8. Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems. MP.7. Look for and make use of structure. MP.8. Look for and express regularity in repeated

Activities

1. Right Triangle Warm Up.

Warm Up	See unit1_9_warmup_righttriangles.doc file (modified from IMP Shadows, pp. 43-44). Students work in groups to answer questions about right triangles such as "Where do you find the hypotenuse?" and "Why must the other two angles of a right triangle be acute — that is, less than 90°?"

2. How Far Away Is That Mountain? The purpose of this activity is to give students an opportunity to explore the ratios between the sides of different right triangles using Geogebra. By starting with a 30° angle, students may see the 1:2 relationship between the opposite side and the hypotenuse.

3. How Far Away Is That Mountain?

Launch	Teacher will show a slide of a mountain far away. Introduce the situation in which the distance to the mountain is known but the

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	height is not. No mirror or shadows are available to work with but a new tool is available: an inclinometer. Teacher can describe the scenario with the inclinometer and the 30° angle. Teacher can launch the activity to find out if there is anything special about the sides of a 30° right triangle.
Group Work	Students can use Geogebra to construct their own right triangles with a 30° angle. They can measure the sides of their triangles and write down their observations. Student Challenges and Misconceptions: Students may find it difficult to organize and make conjectures from the Geogebra activity. Teacher should first encourage students to record the numerical data. Teacher should allow sufficient time for students to experiment with the numbers. If students are still struggling to see the relationship between the numbers, then the teacher can prompt students to look at relationships between particular sides.
Class Debrief	Class can discuss their observations from group work. The discussion should motivate the need for terminology like “opposite” and “adjacent” for describing sides. The 1:2 ratio should surface to describe the opposite / hypotenuse relationship. Teacher will most likely need to prompt for other ratios.
Group Work	Students can work to solve the distance to the mountain using their new discoveries. At this point students will not be on their computers, and they must use their understanding of proportionality to solve.

4. Exploring Other Angles. The purpose of this activity is to give students a chance to explore whether their discovery is only true for 30° right triangle. It will also give students a chance to engage in making and testing conjectures. Students should finish the activity with a clear idea that there are unique ratios for a given angle, but that these ratios hold for all right triangles with that angle (ie all similar triangles).

Launch	Teacher can point out that so far all they know is that these ratios work in right triangles with a 30° angle. Teacher can ask them to make conjectures about what they think might happen in other triangles.
Group Work	Students can use Geogebra to construct right triangles with 4-5 different angles (pre-determined by the teacher) and measure the sides to explore the ratios.

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Very Special Right Triangles

You're familiar with the **isosceles** triangle, in which at least two sides have the same length, and the **equilateral** triangle, in which all three sides are equal in length.

Another special category is the **right** triangle. A right triangle has one angle that measures 90°.

In your group, discuss the following questions. Record your group's thinking.

1. Why must the other two angles of a right triangle be acute—that is, less than 90°?

2. Sketch a right triangle. Then sketch a right triangle with legs twice as long as those of your first triangle.

    a. How does the hypotenuse of the new triangle compare to the hypotenuse of the original?

    b. How do the acute angles of the new triangle compare to the acute angles of the original?

    c. What do your answers to parts a and b tell you about the two triangles?

3. Is it possible for a right triangle to be isosceles? Equilateral? Explain your answers.

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Exploring Right Triangles

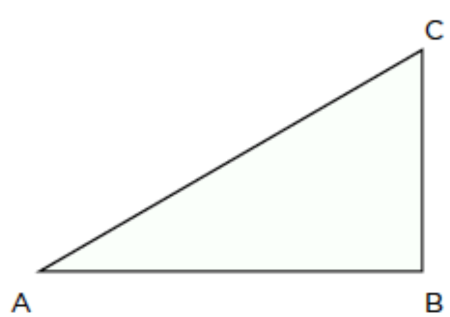
In this activity we will explore relationships in right triangles with particular angle measures.

For each angle, you can experiment and then record your observations.

Although you will be working in pairs to complete this activity, you must each create your own Geogebra triangle. For each angle:

1. Sketch any triangles you and your partner create.
2. Discuss with your partner what you notice.
3. Write down any calculations you make.
4. Write down your observations.

1. Drag the slider to 30°, then move point C. Write down the side lengths. What do you notice?



AB	BC	AC

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Exploring Right Triangles

2. Drag the slider to ______ , then move point C. What do you notice?

3. Drag the slider to ______ , then move point C. What do you notice?

4. Drag the slider to ______ , then move point C. What do you notice?

5. Drag the slider to ______ , then move point C. What do you notice?

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Day 10

Essential Question	Lesson Objective	Standards
How do we use math to find the height of things that are difficult to measure?	Students demonstrate an understanding of trigonometric ratios by constructing values for a trig table. Students use appropriate terminology (sine, cosine, tangent, opposite, adjacent, hypotenuse) while generating trigonometric ratios.	G-SRT.6. Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles. G-SRT.8. Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems. MP.8. Look for and express regularity in repeated reasoning.

Activities

1. Warm Up.

Group Work	Students practice using right triangle ratios by solving a missing side in triangles that include angles used in the previous day.

2. Teacher Talk.

Teacher Talk	Teacher briefly introduces the sine, cosine, and tangent terminology and notation to describe the ratios that they've been working with. These ratios are recorded on a large piece of paper (for later use in a trig table) or a google document.

3. Creating Trig Tables for Sine, Cosine, and Tangent. The purpose of this activity is to give students an opportunity to practice using the new notation and terminology while constructing trigonometric ratios in a trig table.

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Launch	Teacher can ask for ratios for a variety of angles with integer values that different from the day before, motivating the need for a trig table so they'll only have to figure out the ratio once.
Group Work	Students can work in groups to contribute to a trig table for the class and record their findings on a large poster or a Google spreadsheet document.
Class Debrief	Teacher can facilitate discussion about patterns they notice in the table. Possible observations (interesting, but not essential): • This might surface the relationship between sine and cosine for complementary angles. • All values are <1. • Tangent doesn't exist for 90 degrees.

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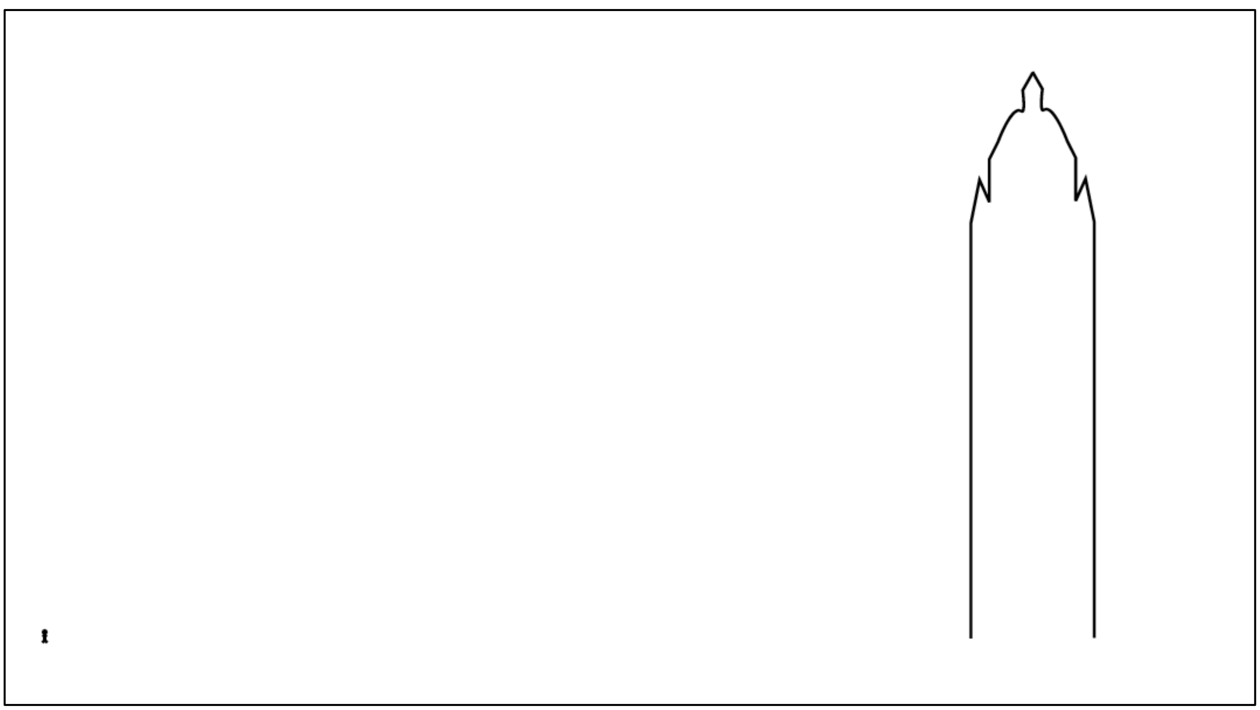

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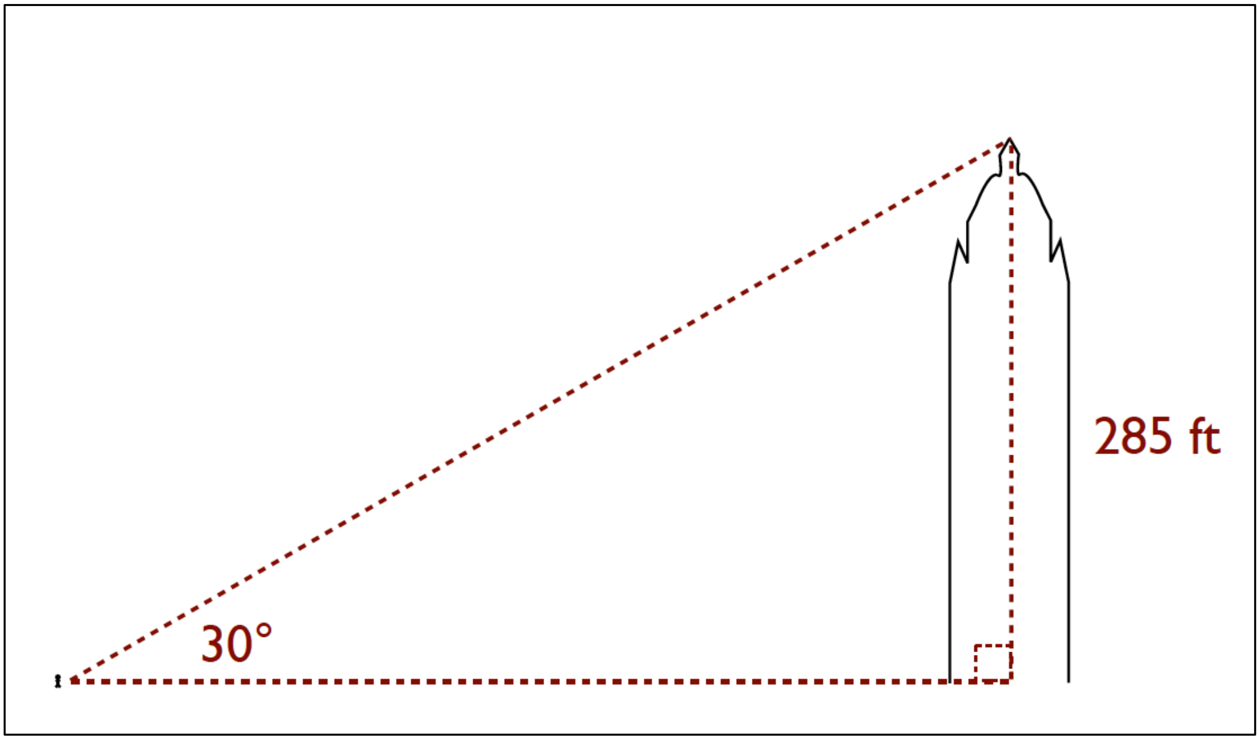

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Very Special Triangles

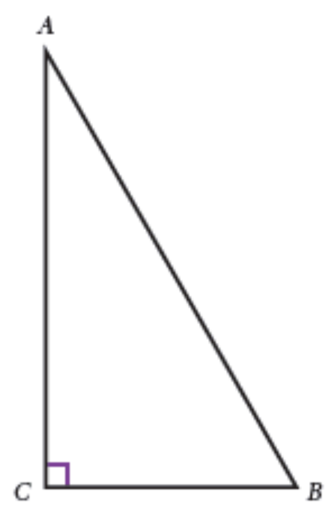
You saw in *Why Are Triangles Special?* that triangles are a special category of polygons. Some triangles are even more special than others.

You're familiar with the isosceles triangle, in which at least two sides have the same length, and the equilateral triangle, in which all three sides are equal in length.

Another special category is the **right triangle.** A right triangle has one angle that measures 90°.

1. Why must the other two angles of a right triangle be **acute**—that is, less than 90°?

Triangle ABC is a right triangle with a right angle at vertex C. The small square inside that vertex is a standard symbol that indicates a right angle.



The two sides of the triangle that form the right angle, $\overline{AC}$ and $\overline{BC}$, are called the **legs** of the triangle. The third side (opposite the right angle), $\overline{AB}$, is called the **hypotenuse.** These two terms apply only to right triangles, not to triangles in general.

Each acute angle of a right triangle is formed by the hypotenuse and one of the legs. For example, $\angle A$ is formed by the hypotenuse, $\overline{AB}$, and by the leg $\overline{AC}$. The leg that helps form an acute angle is said to be **adjacent to** that angle.

For example, $\overline{AC}$ is the leg (or side) adjacent to $\angle A$. That same leg is said to be **opposite** the other acute angle. For example, $\overline{AC}$ is the leg opposite $\angle B$.

continued

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2. What statements can you make about how the lengths of the sides of a right triangle compare to each other? Explain your reasoning.

3. Draw a right triangle, measure the legs, and then draw a right triangle with legs twice as long as those of your first triangle.

   a. How does the hypotenuse of the new triangle compare to the hypotenuse of the original?

   b. How do the acute angles of the new triangle compare to the acute angles of the original?

   c. What do your answers to parts a and b tell you about the two triangles?

4. Draw a right triangle in which the acute angles are different sizes. Is the longer leg opposite or adjacent to the larger of the acute angles? Do you think this is true for all right triangles?

5. Is it possible for a right triangle to be isosceles? Equilateral? Explain your answers.

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Day 11

Essential Question	Lesson Objective	Standards
How do we use math to find the height of things that are difficult to measure?	Students will know how to use their calculators to produce trigonometric ratios. Students will see the relationship between the calculator result and the results from the class produced trig table. Students will practice using their calculators while solving application problems.	G-SRT.6. Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles. G-SRT.8. Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems. MP.4. Model with mathematics. MP.5. Use appropriate tools strategically.

Activities

1. Warm Up.

Group Work	Students should practice using trig ratios to solve problems using the trig table. The last problem involves a non-integer angle, motivating the need for calculators.

2. Teacher Talk.

Teacher Talk	Briefly introduce how to calculate sine, cosine, and tangent on the calculator. Note: It is important to address the degrees / radians issue here.

3. Trig With Calculators. The purpose of this activity is to give students an opportunity to practice using their calculators to solve trig application problems.

Group Work	IMP Shadows, pp. 71-73. Students can work on trig application problems using this new technology.

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4 Inclinometer Lab.

Launch	Show a picture of inclinometer or an actual inclinometer on a cell phone.
Group Work	Groups can use an inclinometer to calculate the heights of various objects outdoors. At least one of these should be an object that they had calculated the height of in a previous activity. Note: Students will need to account for their own height when using the inclinometer.

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Inclinometer Lab

In this activity you will measure the height of three tall objects using an inclinometer.

1. Guess the height of the object.
2. Draw a diagram of the situation.
3. Measure and record all relevant information.
4. Label the diagram appropriately.
5. Calculate the height of the object.

1. Object: ______________________	2. Object: ______________________
3. Object: ______________________	4. Object: ______________________

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Activity

The Tree and the Pendulum

1. Now that you have been introduced to trigonometry, it's time to look again at how to measure the height of a tree.

   Here are the key facts.

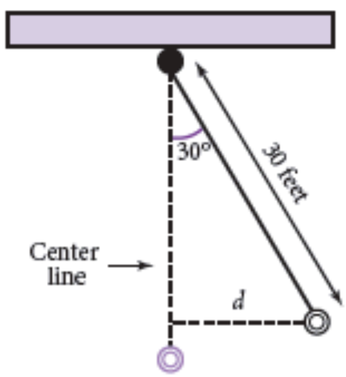
   - Woody is 12 feet from the tree.
   - Woody's line of sight to the top of the tree is at an angle of 70° up from horizontal.
   - Woody's eyes are 5 feet off the ground.

   Describe how Woody could find the height of the tree using trigonometry and these measurements.

2. You can apply trigonometry to the situation from *The Pit and the Pendulum*.

   Suppose a 30-foot pendulum has an initial amplitude of 30°.

   How far is the bob from the center line when the pendulum starts? In other words, what is the distance labeled d?



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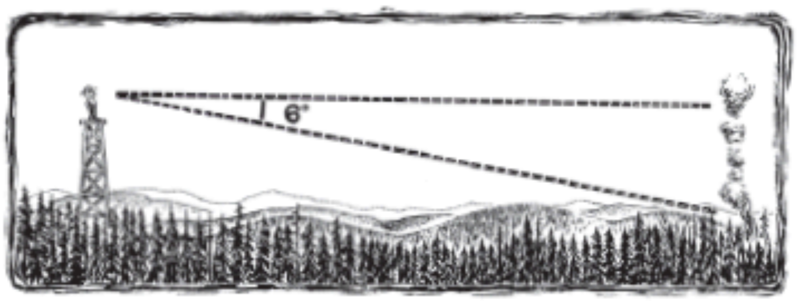
Activity

Sparky and the Dude

1. Sparky the Bear

Sparky the Bear is atop a 100-foot tower. He is looking out over a fairly level area for careless people who might start fires. Suddenly, he sees a fire starting. He marks down the direction of the fire. But he also needs to know how far away from the tower the fire is.

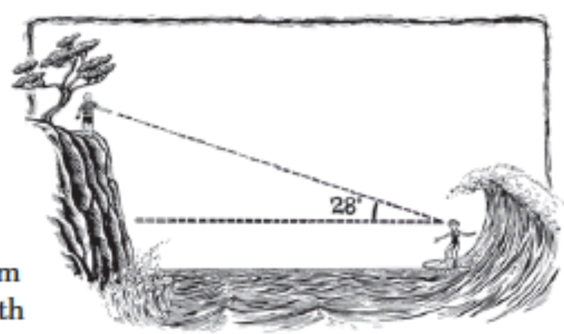
To figure out this distance, Sparky grabs his handy protractor. Because he is high up on top of the tower, he has to look slightly downward toward the fire. He finds that his line of sight to the fire is at an angle of 6° below horizontal. *Note:* This diagram is not to scale.



How far is the base of Sparky's tower from the fire?

2. Dude on a Cliff

Shredding Charlene is out surfing. She catches the eye of her friend Dave the Dude, who is standing at the top of a cliff. The angle formed by Charlene's line of sight and the horizontal measures 28°.



Charlene is 50 meters from the bottom of the cliff. Charlene and Dave are both 1.7 meters tall. They are both 16 years old. The surfboard is level with the base of the cliff.

How high is the cliff?

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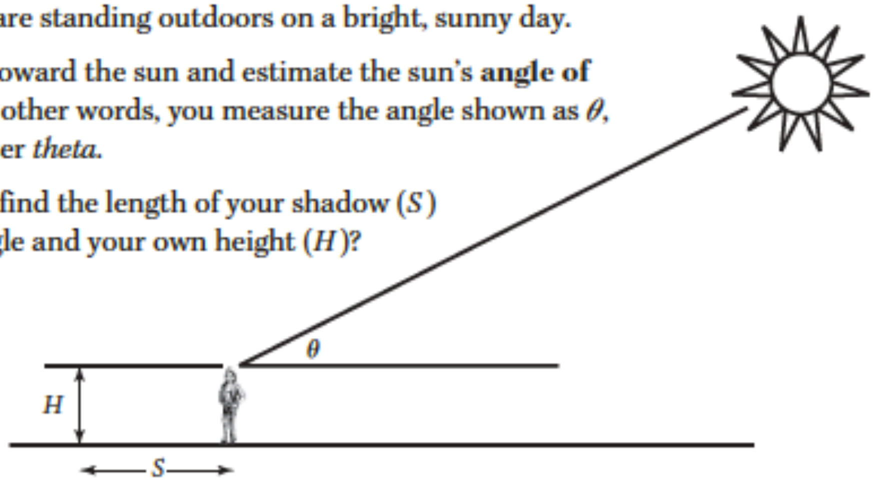
A Bright, Sunny Day

Suppose you are standing outdoors on a bright, sunny day.

You look up toward the sun and estimate the sun's **angle of elevation.** In other words, you measure the angle shown as θ, the Greek letter *theta*.

How can you find the length of your shadow (S) using this angle and your own height (H)?



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Day 12

Essential Question	Lesson Objective	Standards
How do we use math to find the height of things that are difficult to measure?	Students will demonstrate understanding of trigonometric ratios by using them to solve real problems of measurement. Students will be able to explain their mathematical processes both verbally and in writing. Students will compare methods for finding the height of objects. Students will be able to model a real-world situation.	G-SRT.6. Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles. G-SRT.8. Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems. MP.2. Reason abstractly and quantitatively. MP.3. Construct viable arguments and critique the reasoning of others. MP.4. Model with mathematics. MP.5. Use appropriate tools strategically.

1. Inclinometer Lab Posters.

Group Work	Groups can create posters outlining their strategies, procedures, and solutions.
Class Debrief	Students will present their group posters to the rest of the class. Prompts from day 8 (see: unit1_8_gallery walk file) can also be used to assist students in giving feedback. Note: Students may need support in both how to present and how to be in the audience in a productive way. Possible supports might include discussion of norms for listening and questioning or sentence starters for asking questions or giving feedback.

2. Review Game

Review	Facilitate a review game of similarity and trigonometry.

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Day 13

Essential Question	Lesson Objective	Standards
How do we use math to find the height of things that are difficult to measure?	Assessment.	Assessment.

Activities

1. Assessment.

Assessment	IMP Shadows, p. 58, modified. Students will individually work on finding the height of a tree in three different ways. This serves as the assessment of what has been studied so far. The Exemplars Rubric can be used to assess student work.

2. Assessment.

Class Debrief	Facilitate a debrief of the assessment around the pre-selected student work, sharing answers and different strategies. Students can use the Exemplars Rubric to discuss other students' work.

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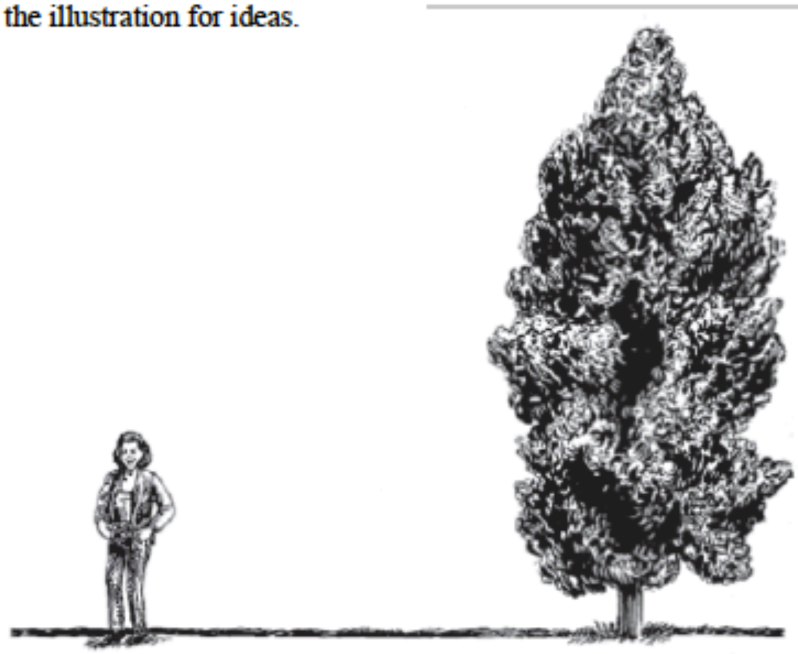
To Measure a Tree

To find the height of a tree, you might climb to the top and drop a long tape measure to the ground while still holding the start of the tape. You'd need a friend on the ground to read off the tree's measurement.

Although straightforward, this method has many difficulties and dangers. Fortunately, there are less hazardous methods.

Your task is to use your knowledge from this unit to describe different methods for measuring a tree. Use the illustration for ideas.



1. Create at least *three* mathematical methods to find the height of the tree.
2. Write down what you'd need to know in each situation.
3. Explain how you'd use that information to figure out the height of the tree.

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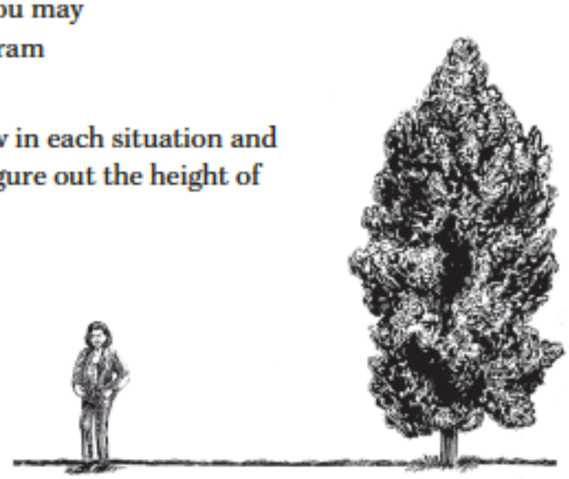
To Measure a Tree

To find the height of a tree, you might climb to the top and drop a long tape measure to the ground while still holding the start of the tape. You'd need a friend on the ground to read off the tree's measurement.

Although straightforward, this method has many difficulties and dangers. Fortunately, there are less hazardous methods, including some that use similar triangles. Your task is to use your knowledge of similar triangles to invent methods for measuring a tree. Use the illustration for ideas.

Create at least three methods, relying on different pairs of similar triangles, to find the height of the tree. Clearly identify the similar triangles in each method. Explain how these triangles are used in your method. You may have to draw extra lines in your diagram to illustrate the situation.

Write down what you'd have to know in each situation and how you'd use that information to figure out the height of the tree.



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Day 14

Essential Question	Lesson Objective	Standards
How do we use math to find the height of things that are difficult to measure?	Students will understand that congruent triangles have congruent corresponding parts (angles and sides). Students will understand that congruency is preserved under rotation, reflections, and translations, but not dilations. Students will discover conditions for triangle congruency (ASA, SAS, and SSS). Students will reason about triangle congruency using ASA, SAS, and SSS.	G-SRT.5. Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures. C-CO.7. Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent. G-CO.8. Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions. MP.1. Make sense of problems and persevere in solving them. MP.3. Construct viable arguments and critique the reasoning of others.

Activities

1. Geogebra Workshop. The purpose of this workshop is to get students ready to use Geogebra in the next activity.

Teacher Talk	Give students a triangle in Geogebra. Teach students how to measure angles and line segments.

2. Congruence Warm Up. The purpose of this warm up is to help students review their understanding of similarity and then begin to consider what it means to be congruent.

Group Work	Students can examine pairs of triangles. They will answer questions such as "Which are similar?" and "Which are exactly the same?" Note: It will be important to make sure that some are

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	congruent but reflected or rotated.
Class Debrief	Groups share conditions that they find for congruency. The discussion should to a definition of congruence-- all corresponding parts (sides and angles) are congruent.

Activities

3. What Is Sufficient? The purpose of this activity is to help students explore what conditions are sufficient in order to guarantee congruence of triangles.

Group Work	Groups work in Geogebra to identify conditions that are sufficient to produce congruent triangles. Groups construct triangles with given specifications (i.e.; A = 20°, b = 4, C = 50°). As a whole class work the first specification together. Do a "think aloud" style approach to how to make and test conjectures for sufficiency. In doing this elicit measures from students in the class, then compare measures for congruency. Students then work in groups to complete #2-8. In groups students compare with peers and observe which conditions produce congruent triangles. Groups work through the entire worksheet before the class debrief. Walk around to individual groups that are having trouble making and comparing triangles. Note: students may have a tendency to try to create congruent triangles with their peers. The teacher may want to encourage students to strive to create different triangles.
Class Debrief	Debrief, summarizing what the class in their investigation discovered. Each of the problems from the worksheet should be covered, to see if those conditions are sufficient "short cuts" to determine triangle congruence. The class should keep a running list of sufficient conditions for triangle congruence.

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Congruent Shapes

Put a check mark next to the pairs you think have exactly the “same shape.” If you and your neighbor disagree, try to come to an agreement. Then, write your thoughts in the Google Form or some other communication tool.

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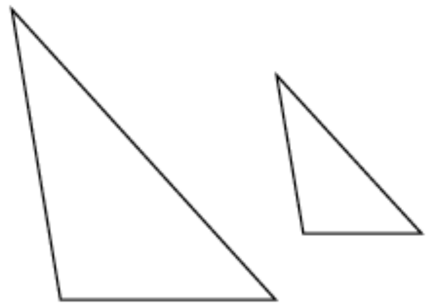

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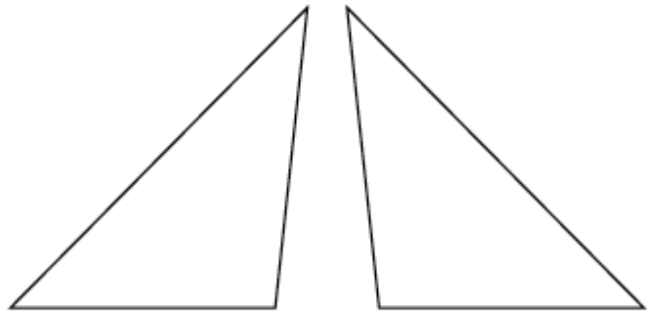

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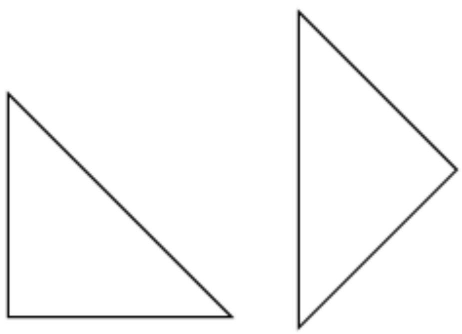

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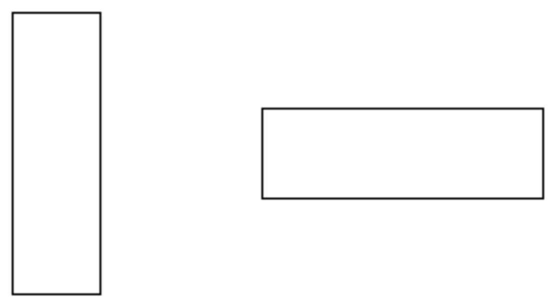

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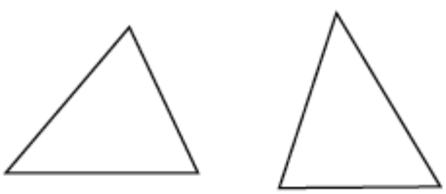

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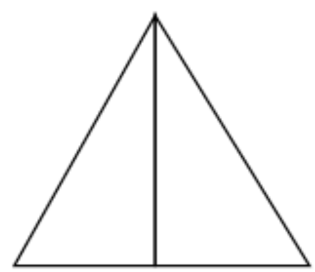

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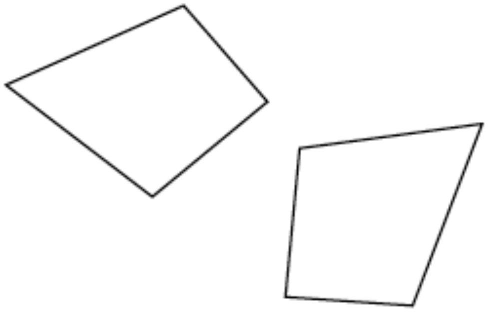

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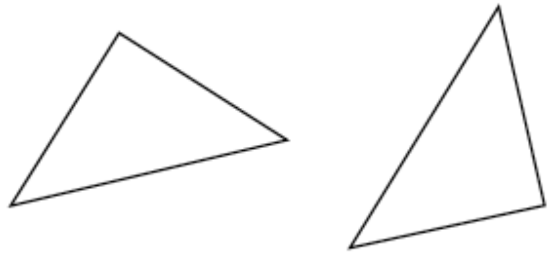

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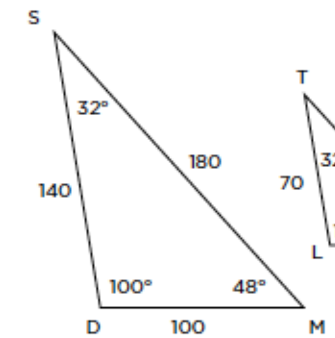
Congruent Shapes

Now develop a rule that will help somebody else decide if two shapes are exactly the same. Be very specific. Write the definition down.

1.



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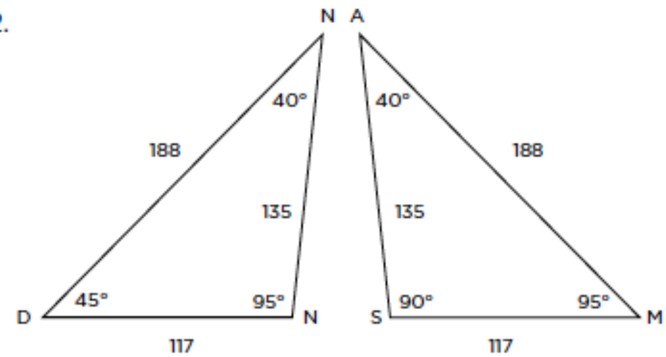

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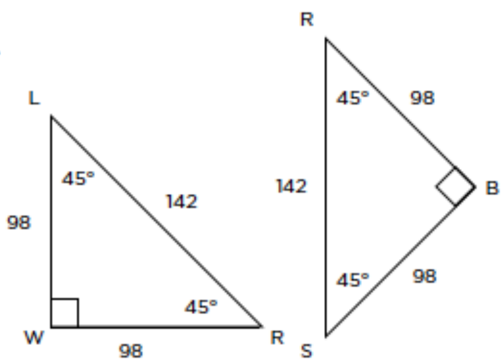

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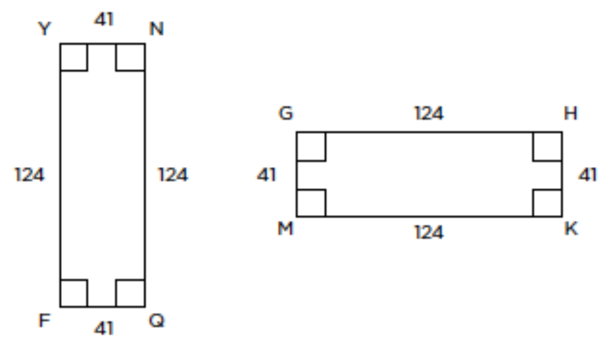

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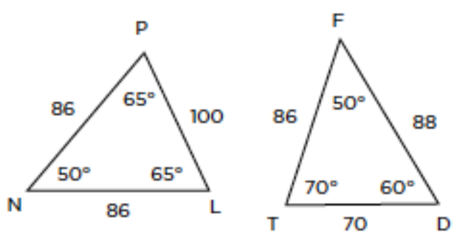

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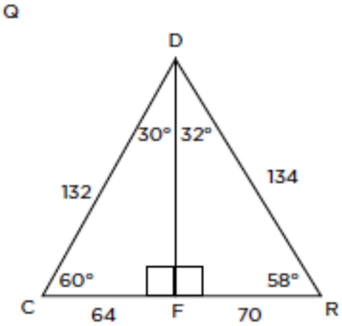

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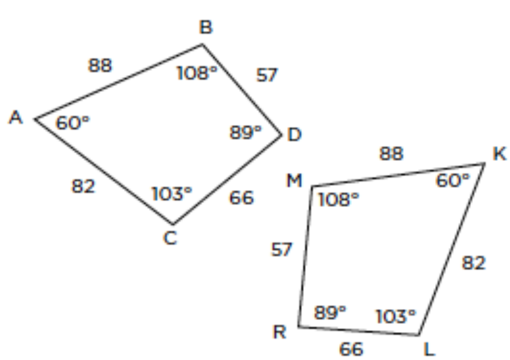

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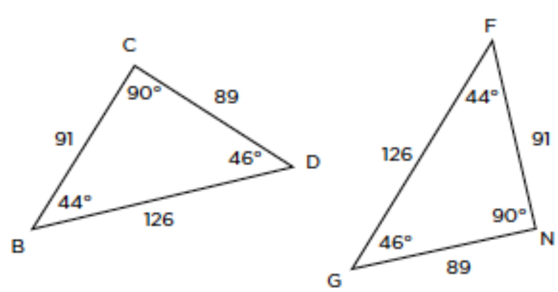

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Triangles: Same Or Di!erent

Make each of the triangles described below. Decide if the information will produce no triangles, exactly one triangle, or more than one triangle. Sketch the possibilities.

1.

Angles	Sides
A = 80°	
B = 60°	
C = 40°	

2.

Angles	Sides
	AB = 6
	BC = 2
	AC = 9

3.

Angles	Sides
A = 20°	AB = 8
	AC = 10

4.

Angles	Sides
A = 110°	AC = 5
B = 50°	

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Triangles: Same Or Di!erent

Make each of the triangles described below. Decide if the information will produce no triangles, exactly one triangle, or more than one triangle. Sketch the possibilities.

5.

Angles	Sides
C = 70°	BC = 5
	AC = 8

6.

Angles	Sides
B = 50°	AB = 7
	AC = 6

7.

Angles	Sides
	AB = 8
	BC = 4
	AC = 10

8.

Angles	Sides
A = 20°	BC = 4
B = 30°	

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Day 15

Essential Question	Lesson Objective	Standards
How do we make a convincing argument?	Students will establish a working definition of what makes a convincing argument.	MP.1. Make sense of problems and persevere in solving them. MP.2. Reason abstractly and quantitatively. MP.3. Construct viable arguments and critique the reasoning of others. MP.4. Model with mathematics.

Activities

1. Driscoll Paper Folding Activity. See Paper Folding Problem handout for students. See Driscoll Paper Folding Section for full activity description.

Pair Work	Students work in pairs to fold the paper and convince themselves that they have achieved the given condition. All pairs should accomplish #1-4. The 5th problem can be an extension for pairs/groups that finish early.
Group Work	Pairs share their folding strategies with their group. Groups will then select one of the folding tasks to create a poster justifying how they know the condition has been met. Each class should have two groups creating posters for the same condition (i.e.; two groups will create poster for paper folding condition #1). Encourage students to explain all thinking and give sufficient evidence for any claims.
Whole Class	Groups present their justifications to the class. Other students will give feedback on presentations: Was there enough information? Were all claims sufficiently backed up? Were any assumptions made? What could have made the argument more convincing?

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3. Class Debrief

Whole Class	Facilitate a discussion about what constitutes a convincing argument. The teacher should create a working document with characteristics/requirements for a convincing argument. This list will be revisited in Day 16. If students finish early work on pile patterns from Day 16

4. Pile Patterns (time permitting)

Group work	If there is time remaining, groups begin to work on the pile pattern warm up from Day 16

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Problem 4

Chapter 2 mentioned the power of paper-folding problems to help students develop the capacity to reason about and with geometric relationships. Paper-folding can also enhance students' understanding of geometric measurement, particularly of area. Several helpful resources, such as Serra (1994), exist to help teachers gather appropriate problems. We have found this problem to be useful, with both teachers and students.

Investigating Area by Folding Paper

For each part of the problem, start with a square sheet of paper and make folds to construct a new shape. Then, explain how you know the shape you constructed has the specified area.

1. Construct a square with exactly ¼ the area of the original square. Explain how you know it is a square and has ¼ of the area.
2. Construct a triangle with exactly ¼ the area of the original square. Explain how you know it has ¼ of the area.
3. Construct another triangle, also with ¼ the area, that is not congruent to the first one you constructed. Explain how you know it has ¼ of the area.
4. Construct a square with exactly ½ the area of the original square. Explain how you know it is a square and has ½ of the area.
5. Construct another square, also with ½ the area, that is oriented differently from the one you constructed in 4. Explain how you know it has ½ of the area.

DVD

The Investigating Area by Folding Paper video clip demonstrates how a group of students explored part 4 of this problem.

Discussion A wide range of GHOM influence comes into play in solving this problem. For example, in navigating parts 2 and 3, the solver needs to be adept at *Reasoning with relationships* to show why triangles are or are not congruent (e.g., in showing the two triangles in Figure 4–20 are congruent).

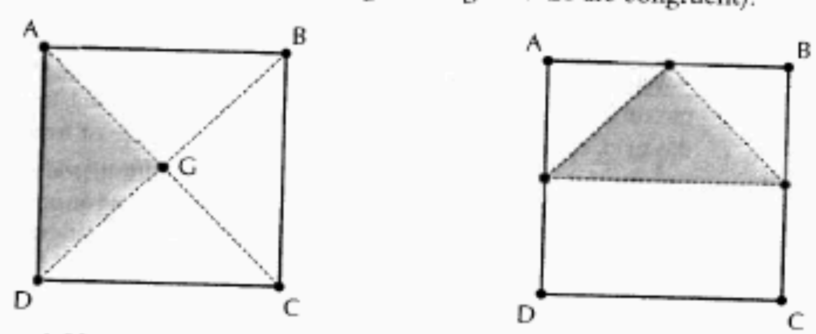

Figure 4–20

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Furthermore, in generating an answer for part 3, one has to think in terms of what is *generally* true when calculating the area of triangles. For example, one might think: "If the square has side 1 and area 1, then for any triangle that fits, $\frac{1}{2} \times$ base $\times$ height $= \frac{1}{4}$. So base $\times$ height is $\frac{1}{2}$." That way of *Generalizing geometric ideas* can lead to folded constructions like those in Figure 4–21. Typically, solvers answer part 4 by folding to make the square shown in Figure 4–22.

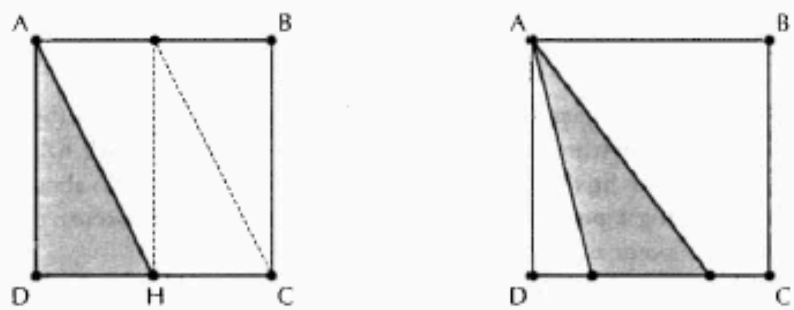

Figure 4–21

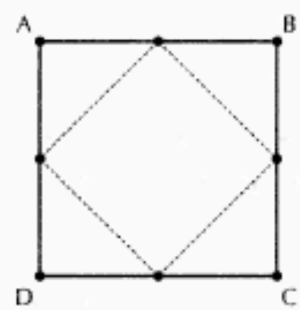

Figure 4–22

In that case, solving for part 5 can involve both *Investigating invariants* and *Balancing exploration and reflection*. The former comes into play in showing that such a square must exist without actually finding it. One can do this by imagining a point E on a diagonal of the square, say diagonal DB, close to the bottom. Connecting E to sides AD and DC with perpendicular segments forms a square (Why?). Moving E along the diagonal toward B is an action that preserves the "squareness" of the figure (Why?) but increases the area. (See Figure 4–23.)

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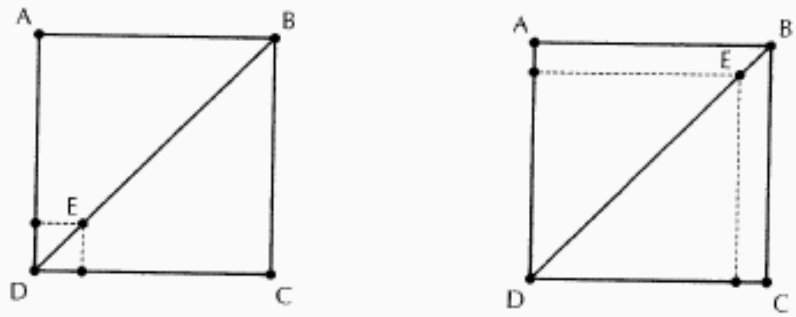

Figure 4–23

In fact, the area of the square increases from 0 to the full area of ABCD. Somewhere in between there must be a square with half the area of ABCD. This fits the indicator of *Investigating invariants:* "thinks about the effects of moving a point or figure continuously and predicts occurrences in between one point and another."

To get the exact square through paper-folding, one would need to pinpoint the size of the side of this square. Here is one way.

For simplicity, let's take the area of ABCD to be 1. Then the square in question has area ½, so must have side $\frac{1}{\sqrt{2}} = \frac{(\sqrt{2})}{2}$.

Where in the larger square might we find a length of $\frac{(\sqrt{2})}{2}$? The diagonal of ABCD is $\sqrt{2}$, so folding the diagonal in half will give a length of $\frac{(\sqrt{2})}{2}$. The following sequence of pictures shows one way to create the square out of this (see Figure 4–24).

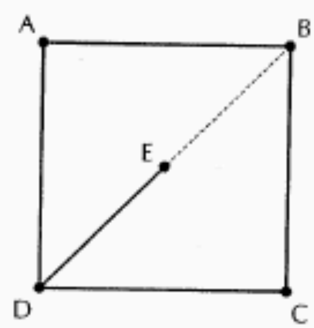

Figure 4–24

Hold D as the pivot point and fold E down to side DC, producing point F (see Figure 4–25).

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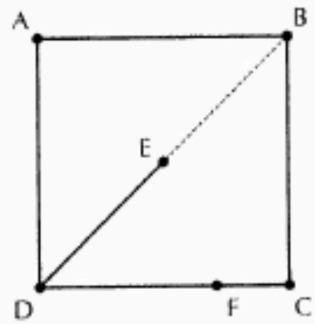

Figure 4–25

Hold D as the pivot point and fold E over to side AD, producing point G (see Figure 4–26).

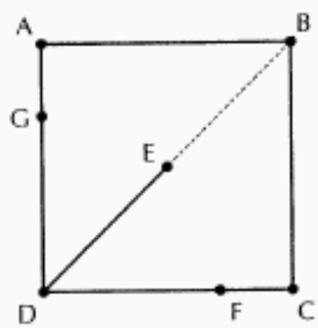

Figure 4–26

Hold F as the pivot point and fold D until it lands on diagonal DB, producing point H (see Figure 4–27).

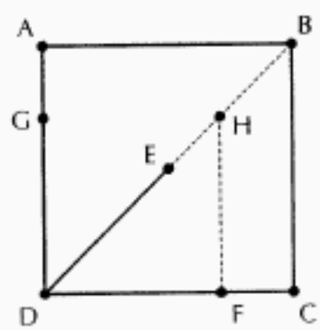

Figure 4–27

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Day 16

Essential Question	Lesson Objective	Standards
How do we make a convincing argument?	Students will be able to critique the arguments of others. Students will construct justifications for the general rule in a linear pile pattern.	MP.1. Make sense of problems and persevere in solving them. MP.2. Reason abstractly and quantitatively. MP.3. Construct viable arguments and critique the reasoning of others. MP.4. Model with mathematics.

Activities

1. Pile Pattern Warm-Up. See Pile Pattern 1 task card. The goal of this warm-up is to allow students to explore and make sense of a pile pattern before they examine justifications for the number of blocks in case n.

Group Work	Students work in groups of 4 on the Pile Pattern Warm-up. If groups reach the generalization early, they should work on writing a convincing argument for why their rule works. Note: Since the next activity presents the students with the correct generalization, it is not important that every group reach a correct and final generalization in the warm-up. The goal is for students to have spent time exploring the growth pattern so that they are better able to critique the arguments of others.

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2. Pile Pattern Proof Critique. The goal of this task is for students to extend their understanding of what makes a convincing argument by critiquing justifications written by others.

Launch	Briefly asks the class to brainstorm a list of what they currently believe makes a convincing argument. This connects with the activities the day before on paper folding. The list can remain on the board during the next part of the activity. Explains that several students have constructed justifications for the pile pattern problem generalization 2n+1. "Your task is critique each argument. You should be thinking about what makes each argument convincing or not convincing".
Group work	In small groups, students begin working on the pile pattern justification critique activity. Two options for implementation: (1) You can ask students to rank the justifications first (by sorting) and then complete the written sections or (2) ask students to go through the packet, in sequence, filling out each page first before ranking them in order of least to most convincing. Teacher Talking Points • Playing the skeptic. Remind students to play the skeptic and try to find fault with each argument.
Debrief	Pre-select groups to share out their critiques of the six different arguments. Other students should be ready to agree or disagree with the presenting group's critique. Background of the task The six justifications are explicitly designed to target classic types of student justifications of varying levels of rigor. • (Annie) Proof by external authority. In this case, students refer to an external authority to justify what it works. • (Jonathan) Proof by restatement of the conjecture. • (Dan) Proof by cases. In this type of proof, students show that the generalization works by checking a single case. • (Kathy) Proof by many cases. Similar to the first type, this proof uses many cases to show that it works. • (Sarah Kate) Proof by generalized example. In this case, a student shows that the rule works with a single example but treats it in a general way by making an argument about the structure. • (Jo) Deductive proof. This example is the most rigorous, since it makes an argument about the growth pattern in general rather than relying on cases.

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	Once groups have shared their critiques, ask the students to revisit their list of characteristics of a convincing argument.

3. Constructing Pile Pattern Justifications. See Pile Pattern 2 task card. The goal of this warm-up is to allow students to try constructing their own convincing arguments for a different pile pattern.

Group Work	Students work in groups of 4 on Pile Pattern 2. They first construct a generalization and then construct a justification for their general rule. This can serve as a formative assessment for their initial understanding of what makes a convincing argument.

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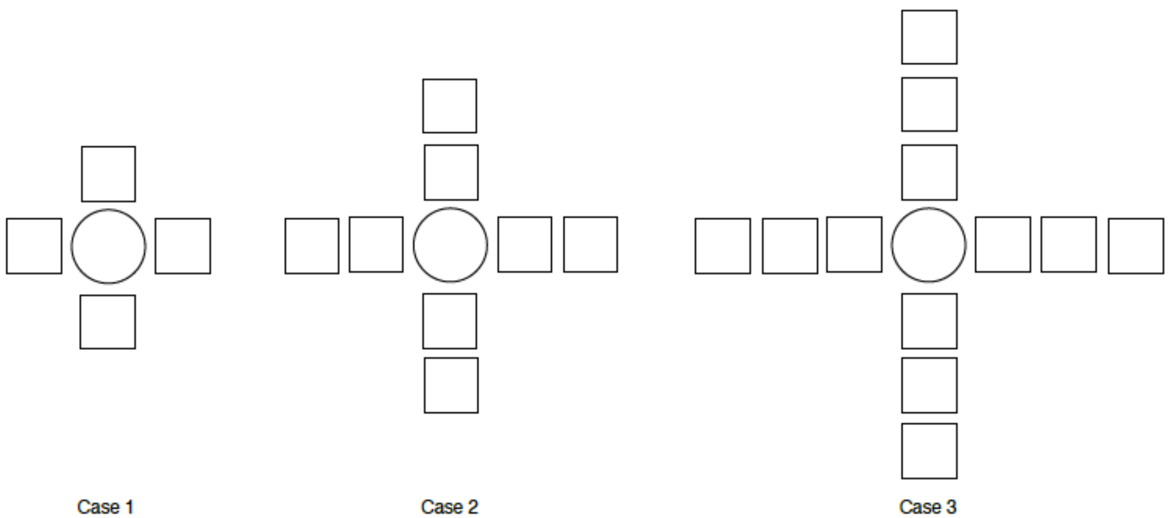
Pile Pattern Problem 2



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1. How many blocks will there be in case 4? Case 5? How do you know?

2. What do you think case 0 looks like? How do you know?

3. How many blocks do you think there will be in case 100? Explain your thinking.

4. Describe how to find the number of blocks in any case.

5. Write a general rule to find the number of blocks in case n.

6. Write a justification (a convincing argument) for how you know that your rule works for any case.

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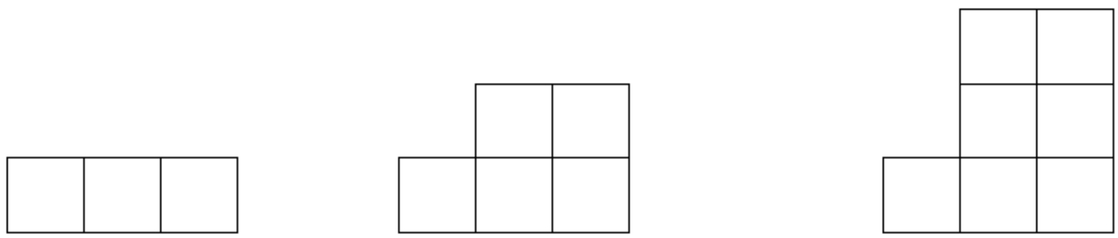
Pile Pattern Problem 1



Case 1 Case 2 Case 3

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1. How many blocks will there be in case 4? Case 5? How do you know?

2. What do you think case 0 looks like? How do you know?

3. How many blocks do you think there will be in case 100? Explain your thinking.

4. Describe in words how to find the number of blocks in any case.

5. Write a general rule to find the number of blocks in case n.

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6. How you know that your rule works for any case?

Annie's Proof

Everyone else in my group got $2n+1$ so it must be true.

What do you find convincing about this proof?

Be skeptical. What did not convince you in this proof? What did you feel was missing or unclear?

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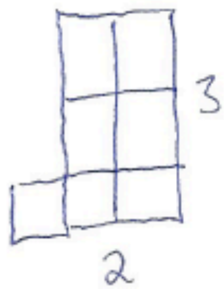
Dan's Proof



The rule is $2n+1$.

$2 \cdot 3 + 1 = 7$

So the rule works.

What do you find convincing about this proof?

Be skeptical. What did not convince you in this proof? What did you feel was missing or unclear?

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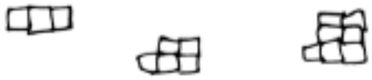
Kathy's Proof

Case: 1 2 3



Case	# blocks
1	3
2	5
3	7

I drew each case, then I checked to see if the rule worked for each case:

$2(1)+1=3$ ✓

$2(2)+1=5$ ✓

$2(3)+1=7$ ✓

The rule worked, so the rule is always true.

What do you find convincing about this proof?

Be skeptical. What did not convince you in this proof? What did you feel was missing or unclear?

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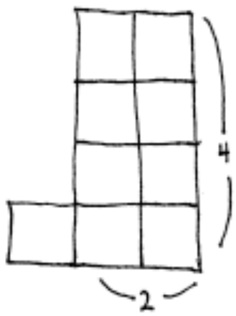
Sarah Kate's Proof



There are two blocks in each row, except for the bottom one, which has one extra.

There are 4 rows because it's case 4.

So $4 \cdot 2 = 8$ is the number of blocks in all the rows and you have to add one more for the bottom row

$$2 \cdot 4 + 1 = 8 + 1 = 9$$

What do you find convincing about this proof?

Be skeptical. What did not convince you in this proof? What did you feel was missing or unclear?

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Jonathan's Proof

2n+1 is the rule since there are always that many blocks when you count them up.

What do you find convincing about this proof?

Be skeptical. What did not convince you in this proof? What did you feel was missing or unclear?

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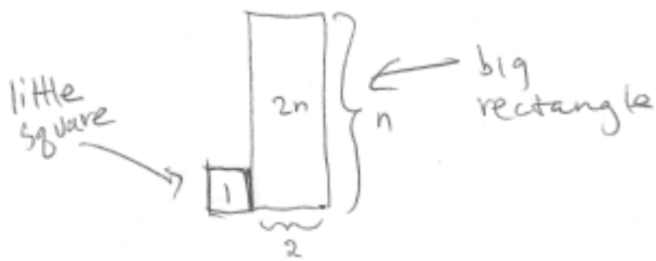
Jo's Proof

You can break every case into two shapes:



The big rectangle has 2 blocks per row, and there are n rows total.

This means there are 2n blocks in the rectangle.

Since there is always one more block from the little square, the total number of blocks must be:

$2n+1$

What do you find convincing about this proof?

Be skeptical. What did not convince you in this proof? What did you feel was missing or unclear?

Day 17

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Essential Question	Lesson Objective	Standards
How do we make a convincing argument?	Students will find the area of an equilateral triangle with a given side length. Students will justify each step in the process of finding the area.	MP.1. Make sense of problems and persevere in solving them. MP.2. Reason abstractly and quantitatively. MP.3. Construct viable arguments and critique the reasoning of others. MP.4. Model with mathematics.

Activities

1. Finding the Area of an Equilateral Triangle. The purpose of this activity is to give students a chance to find the area of an equilateral triangle. Students must justify each step in their process.

Pair Work	In pairs students work to find the area of a given equilateral triangle (see worksheet). Note: Possible solution strategies: 1) Using trigonometry and 2) Using Pythagorean theorem.
Group Work	Pairs share their work with their group of four. The group then decides which method(s) they wish to present to the class. Groups make a poster showing how they found the area of the triangle. Students must give a reason for why each step is valid.
Whole Class	Preselect 2-3 groups to present their posters to the class. Select groups to present on different solution strategies. As students present, other students will give feedback on presentations: Was there enough information? Were all claims sufficiently backed up? Were any assumptions made?
Debrief	As a class discuss summarizing the various solution strategies to finding the area of an equilateral triangle.

2. Proving Triangle Congruence. The purpose of this activity is to give students a chance to

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prove that triangles are congruent.

Group work	In groups students write out proofs for congruent triangles. Students examine pairs of triangles, including triangles that have shared sides or angles. At this point there is no formal structure for writing proofs, so students should document their rationale however they see fit. Encourage students to make sure that their justifications meet the "what makes a convincing argument" criteria.
Group Presentation	Ask each group to present to the class a proof of a specific pair of triangles (designated by the teacher). The class will be responsible for giving feedback on the different presentations, with an emphasis on whether the proof was convincing (as compared to the working definition of convincing argument). The conversation should also elicit other potential ways to prove. If possible, use of a document camera would be useful to show proofs.

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Day 18

Essential Question	Lesson Objective	Standards
How do we use math to find the height of things that are difficult to measure?	Students will develop their understanding of parallelograms alongside their understanding of proofs.	MP.3. Construct viable arguments and critique the reasoning of others.

Activities

1. Angle Chase.

Warm Up	Students review their knowledge of the angles surrounding transversals, parallel lines, and triangles using an angle chase.

2. Geogebra Proof Lab.

Group Work	Students will explore a parallelogram. Their goal will be to use the tools that measure angles and sides and draw conjectures about the properties of a parallelogram. (ie. "Opposite angles look congruent.") They should feel free to draw diagonals or other auxiliary lines.
Class Debrief	As a class list all of the conjectures. Compile those into a class list. If anybody makes a conjecture that seems incorrect to another group other groups should feel free to point out counterexamples.
Group Work	Groups no prove a different conjecture each. Provide a minimal amount of scaffolding. If they are completely stuck, encourage them to create triangles and look for congruency. Groups design posters to present their proofs.
Class Debrief	Groups present their proofs. Remind the other students to be critical. They must not let any sloppy proof logic slip by. If, in the course of proving one conjecture, the students use congruent parts to prove another conjecture, that should get written on the list of conjectures. Proven conjectures should get some kind of large check mark for validation.

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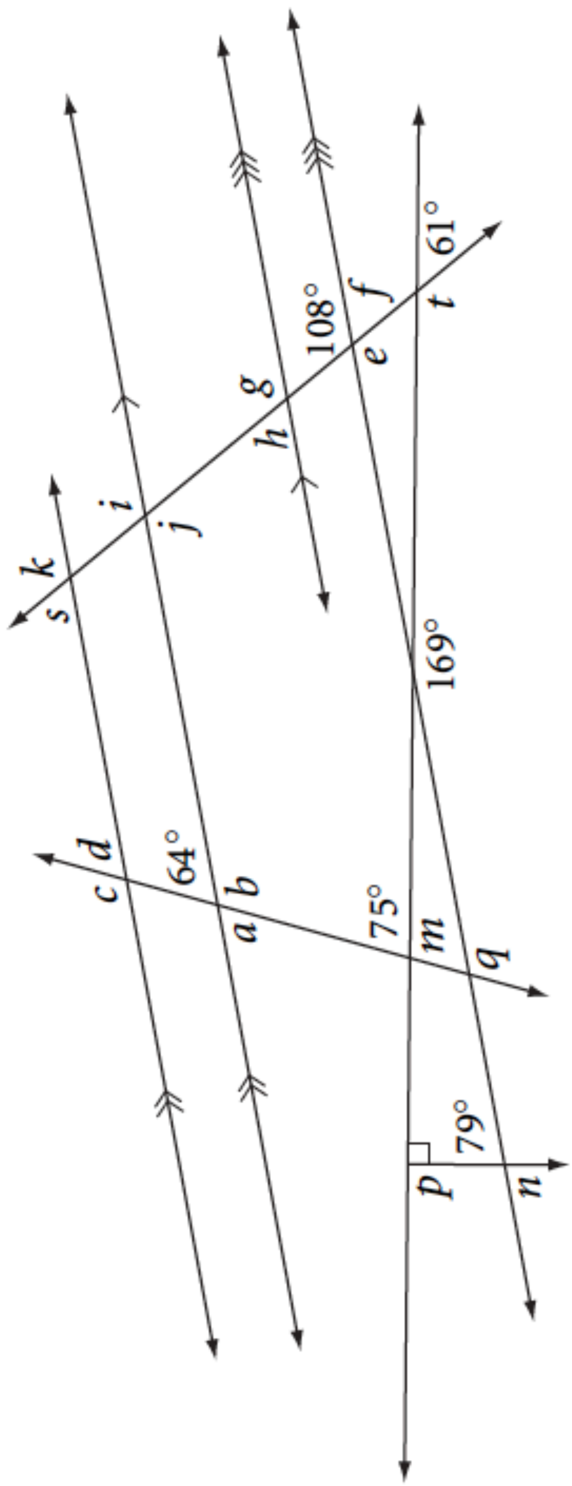
Exercise 7



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Day 19

Essential Question	Lesson Objective	Standards
How do we use math to find the height of things that are difficult to measure?	Students will develop their understanding of parallelograms alongside their understanding of proofs.	MP.3. Construct viable arguments and critique the reasoning of others.

Activities

1. Forms of Proof.

Teacher Talk	Select a proof from the previous day and demonstrates how it can be formed using a paragraph, two-column, or flowchart proof.

2. Geogebra Proof Lab. This is a similar lab to Day 19's, only now they're given a rhombus, a trapezoid, a kite, a square, and a rectangle instead of a parallelogram.

Group Work	Students make a conjecture about each of those shapes. Then they prove it using the forms of proof described in teacher talk. Each form must be used once. Students create a poster from their proofs and post them around the room.
Gallery Walk	Students walk around and read everyone else's proofs. Everyone's goal is to make sure that everyone else's proofs are fully justified.
Class Debrief	Teacher and students discuss the proofs they didn't feel were fully justified. Once everyone is comfortable with the proofs, the compile them into a list for distribution and use on day 20.

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Day 20

Essential Question	Lesson Objective	Standards
How do we make a convincing argument?	Students use properties of quadrilaterals to solve problems.	MP.1. Make sense of problems and persevere in solving them. MP.2. Reason abstractly and quantitatively. MP.3. Construct viable arguments and critique the reasoning of others. MP.4. Model with mathematics.

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Day 21

Activities

1. Using Properties of Quadrilaterals. The purpose of this activity is to give students a chance to use the properties of quadrilaterals that were conjectured and proved on the previous day. Since groups only worked on a single type of quadrilateral, this activity gives them a chance to work with more properties and types of quadrilaterals.

Group Work	Students work in groups on the Quadrilaterals task card. They should have access to a list of the properties that were proved about quadrilaterals in the previous day.

2. Justifications and Proof Assessment. This assessment examines students' proficiency in two areas: (1) Constructing geometric proofs and (2) Critiquing the justifications of others.

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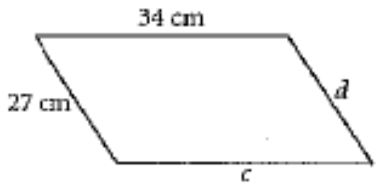
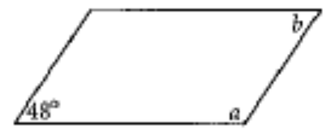
Quadrilaterals

In questions 1-6, each figure is a parallelogram.

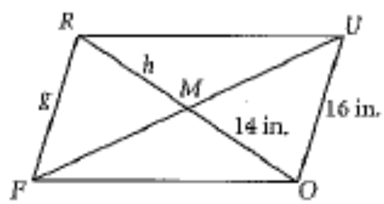
1\. Find the values of c and d.



2\. Find the values of a and b.

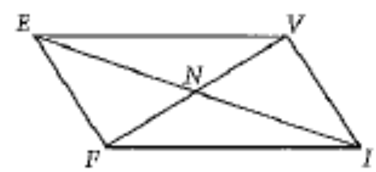


3\. Find the values of g and h.

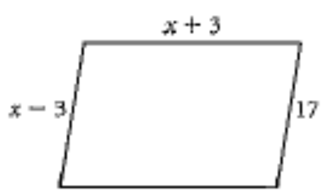


4\. $VF = 36$ m $\quad EF = 24$ m $\quad EI = 42$ m

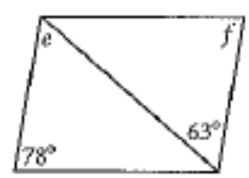
What is the perimeter of ΔNVI?



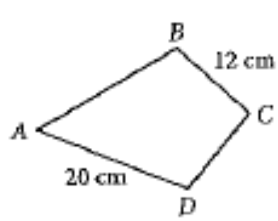
5\. What is the perimeter?



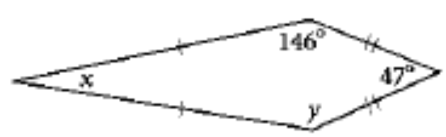
6\. Find the values of e and f.



7\. $ABCD$ is a kite. Find its perimeter.



8\. Find the values of x and y.



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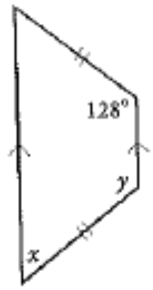
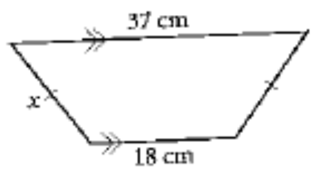
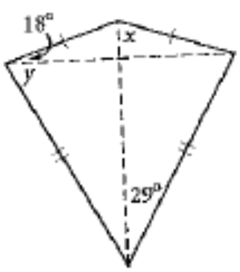
9. Find the values of x and y.



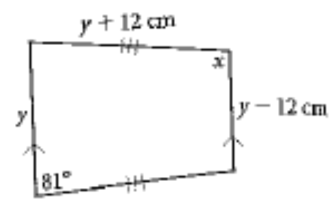
10. The perimeter of this shape is 85 cm. Find the value of x.



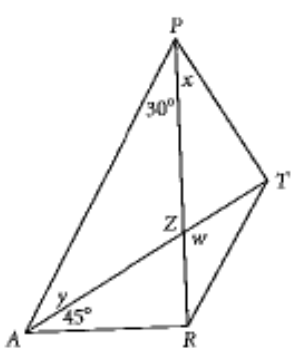
11. Find the values of x and y.



12. Find the values of x and y.

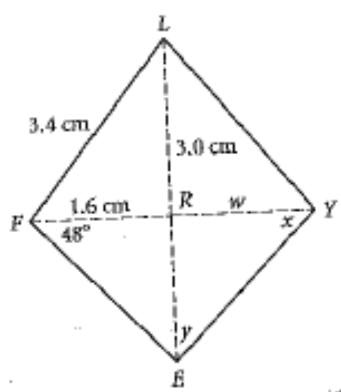


13. *ARTP* is an isosceles trapezoid with $RA = PT$. Find the values of w, x, and y.



$w = 120°, x = 45°, y = 30°$

14. *FLYE* is a kite with $FL = LY$. Find the values of w, x, and y.



$w = 1.6$ cm, $x = 48°, y = 42°$

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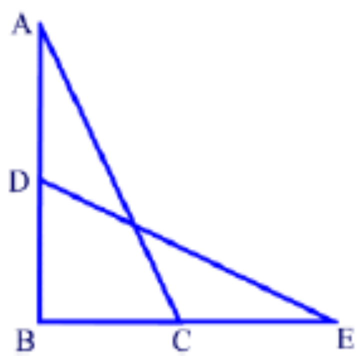
Justification and Proof Assessment

1. In the figure below,



a. Name a pair of congruent triangles in the figure.

b. Write a proof for how you know that the triangles in part (a) are congruent. You can choose which form of proof (paragraph, flow-chart, two column) to use.

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2. Sarah and Kathy are both trying to prove that the sum of the interior angles in n-sided polygon is 180°(n - 2).

Sarah's Proof	**Kathy's Proof**
A square has 360 degrees and 4 sides. You can put 4 into the equation. 180°(4 - 2) = 180°(2) = 360°. So the rule works.	If you draw a polygon, you can make triangles inside the polygon by drawing lines. *Figure with triangles drawn in.* Every triangle has 180 degrees. So you add up all of the triangles and you get 180°(*n* - 2).

a. Which proof do you find more convincing? Why?

b. Pick one student. Write her a letter with some suggestions for how she could make her proof more convincing.

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