PACKERS' PUZZLE

Teacher's Guide — Getting Started

Kai Chung Tam
Macau, People's Republic of China

Purpose

In this two-day lesson, students consider ways to estimate the number of spheres that will fit within a container. They also will try to pack as many as possible into differently shaped containers.

The objective of this lesson is to have students use geometric solids so that they can solve basic packing problems that arise in the real world.

Prerequisites

It is assumed that students are familiar with the calculation of area and volume of various shapes. Other geometrical concepts related to circles, such as radius, diameter, and tangent lines, are also relevant. Informal exposition of rigid motions (parallel translation, rotation, and reflection) is preferred.

Materials

Required: Calculator, circular tokens of various sizes (e.g., pennies, bottle caps, checkers), and two-dimensional "containers". As a preliminary step, teachers need to prepare photocopies of 3 shapes (squares, circles, and equilateral triangles) of three different sizes each. Be sure to note the measurements (sides and radii) of each of these shapes and for each token for students to make proper calculations.
Suggested: None.
Optional: Digital scale. A jar of candy or any container of identical objects.

Worksheet 1 Guide

The first four pages of the lesson constitute the first day's work. Initially students can work on the first two pages individually, but for the next two pages, they should to be organized into groups. Each group will be provided tokens and shapes (containers) to model the orange packing situation. By combining different ideas that the students came up with before they were separated into groups, they can fill out the table provided on the third page of the lesson and answer the questions that follow.

Worksheet 2 Guide

The fifth and sixth pages of the lesson constitute the second day's work in which students should realize that a dense packing is wanted. After a discussion of the density and the unit of a regular arrangement in the plane, students will calculate the theoretical density of rectangular and hexagonal arrangement and compare the result with the first worksheet. Finally, students will think about the extension to three dimensions.

CCSSM Addressed

G-GMD.3: Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.
G-MG.1: Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder).
G-MG.2: Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot).
G-MG.3: Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).

PACKERS' PUZZLE

Student Name:_____ Date:_____

At the Orange Festival right after a bumper crop, a farmer invites guests from all over the town. He shows the guests a full box of randomly arranged oranges, stating that anyone who could guess the exact number of oranges can take home as many oranges as he or she can carry.

© Wilfred Stanley Sussenbach | Dreamstime.com

Is there a difference between the randomly packed oranges on the left and the regularly arranged oranges on the right?

If you cannot just pour out all of the oranges and then count them one by one, what technique would you use to determine correct the number of oranges in the box?

Leading Question

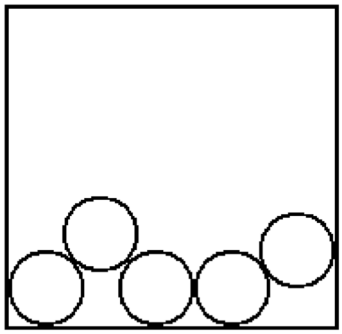
How can you determine the correct number of oranges that are in the container?

PACKERS' PUZZLE

Student Name:_____ Date:_____

1. Before you can answer the question, you need to make some assumptions to think more effectively. One assumption you can make is that all the oranges are spheres. What other assumptions could you make in your model that might not be true in the real world, but are basically useful in creating a mathematical model?

2. Often it is simpler to look at an easier question before trying to attempt a difficult one. In a two-dimensional model, containers become planar. For example, they can be rectangles or triangles. Oranges become circles, which cannot intersect with each other or with the container. What methods might you use to estimate how many circles can be packed into a box, without direct counting? Describe how one of your methods works using words and mathematical notation.



3. How could you use your knowledge of the area of circles to determine the maximum number of circles that can fit into your container?

PACKERS' PUZZLE

Student Name:_____ Date:_____

With your group, use the containers (shapes) that your teacher has provided and fill them with your oranges (tokens). Try different size shapes and tokens. For each trial, choose one shape and one type of token, then try to fit the tokens into the shape. Describe how you fit them in, and fill in one row of the following table. Try to have five unique trials.

Group Names:					Shape of Container:			
#	Side or radius of container (cm)	Capacity of container C (cm^2)	Radius of token r (cm)	Area of token (cm^2)	Maximum number N'	Actual token fit N	$N' - N$	Density $N\pi r^2/C$ (%)
1								
2								
3								
4								
5								

4. How did you fill in your tokens in each case?

5. What accounts for the difference $N' - N$?

6. How could you improve the estimation N', so that $N' - N$ becomes smaller?

When you propose your way of estimating N', think of question 5.

PACKERS' PUZZLE

Student Name:_____ Date:_____

7. The density column calculates the true total area of the tokens, which is $N\pi r^2$, divided by the capacity, C, of the shape. What kind of arrangement would give you a higher or lower density, a random arrangement or a regular arrangement? Why?

8. What other ways of calculating the number of oranges might exist? How do you think the farmer knows the number? Do you think he actually counted all of them?

How might measuring the weight of an orange help you to determine how many are there?

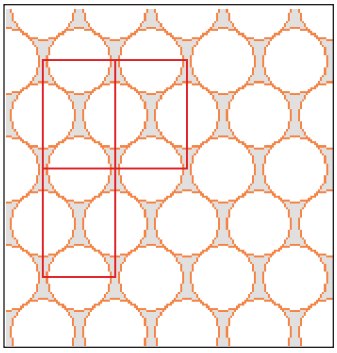
9. Should your answer be a whole number? Explain your reasoning.

PACKERS' PUZZLE

Student Name:_____ Date:_____

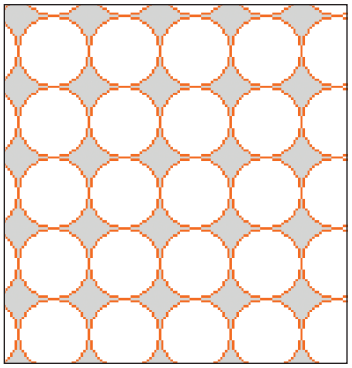
Recall from the previous lesson that the prize for winning was that you got to take as many oranges as you could carry. Suppose that you have already won the prize and the farmer offers some boxes to use to pack oranges.

A **regular arrangement** is one that can repeat indefinitely and looks the same wherever you see it. More precisely, there is a **unit** of arrangement so that you can do parallel translations to repeat the pattern in any direction. The figure on the right shows a regular arrangement, and indicates three copies of the unit. Using just one unit repeatedly, you can extend the picture as far as you want.

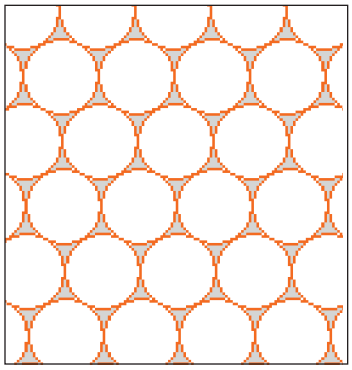


10. Find and draw a unit in each the following two arrangements.

A:



B:



11. If you have a container, **density** is the area used divided by the total area of the container. Find the density of the two units that you have chosen for arrangement A and arrangement B, as if the unit is a container.

PACKERS' PUZZLE

Student Name:_____ Date:_____

12. How do the two densities differ? Can you say that the density within one unit represents the overall density? Why or why not?

13. Compare the results to your classmates' and look at the unit that they have chosen. Did you choose the same unit? Did you get the same density?

14. The Arrangement A is a **square arrangement**, while arrangement B is a **hexagonal arrangement**. Do you understand why they are named this way? Why do you think they were named this way

15. If you have enough identical spheres (e.g., oranges, gumballs, baseballs), try to pack them regularly into a container for which you know the volume. Knowing what you now know about density, what are possible arrangements? What is the density of each arrangement according to experiment or calculation?

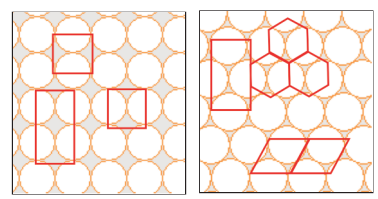
PACKERS' PUZZLE

Teacher's Guide — Possible Solutions

The solutions shown represent only some possible solution methods. Please evaluate students' solution methods on the basis of mathematical validity.

1. Some examples: all oranges are identical objects, all of them are spheres, the spheres and the container should not overlap each other, and the spheres and the container are rigid.
2. Any method will do, but good methods will have some form of organization to them. For example, organize the tokens in rows and stack the second row on top of the first, so the circles are notched together, and the height is minimal.
3. Students should use the formula for the area of a circle, $A = \pi r^2$, and the area of the shape (capacity) to determine an upper bound for the number of oranges that can fit inside the shape by dividing the capacity by the area of one token.
4. See question 2 for an answer. Since the students are now in groups, combinations of methods might have also been created.
5. The space not occupied by the circles accounts for the difference. The more unused space there is, the larger the difference.
6. If it is a regular arrangement, the denominator can be changed to the area of a "unit containing one circle". If it is a random arrangement, it is not very easy to estimate well by this method; however, in three dimensions there is a way to estimate the volume of the unused space. Use any liquid to fill it up the container to capacity and then measure the amount of liquid used! These are not advanced methods so students should be motivated to find one.
7. Some students might pack tokens into the shape randomly while others might do a regular arrangement, so the "density" will vary. Yet, if we fix one arrangement of packing (random, squared, hexagonal), the density has only a little difference.
8. One other way to calculate is dividing the total weight by the weight of one orange.
9. According to our assumption "that all oranges are equal", the quotient should be exactly the same as the number of oranges, but in reality, sizes vary.

10–15.



In a square arrangement, each circle touches four other circles; in a hexagonal arrangement, each circle touches six others. We use red lines to draw a unit. On the left, all these rectangles are correct units. Spheres have diameter equal to 1 cm. In the two squared ones, the total area of a unit is 1 cm^2. The used area is $\pi \cdot (0.5\text{ cm}^2)$ therefore the density within each unit is $\frac{\pi(0.5\text{ cm})^2}{1\text{ cm}^2} = \frac{\pi}{4} \approx 78.54\%$. The larger rectangle gives the same density. On the right, the rectangle, parallelogram, and hexagon are all correct "units". For the rectangle, total area $=(1\text{ cm})\cdot(\sqrt{3}\text{ cm}) \approx (\sqrt{3}\text{ cm}^2)$, and the used area = area of 2 circles $= \frac{\pi}{2}$ cm^2, therefore the density $= \frac{\pi}{2\sqrt{3}} \approx 90.69\%$. Using the parallelogram, the total area becomes $(1\text{ cm}) \cdot (\frac{\sqrt{3}}{2}\text{cm}) = \frac{\sqrt{3}}{2}\text{cm}^2$, and the used area equals exactly one circle, therefore the density is $\frac{\pi}{4} \div \frac{\sqrt{3}}{2} = \frac{\pi}{2\sqrt{3}}$, the same as before. The hexagon also gives the same result.

PACKERS' PUZZLE

Teacher's Guide — Extending the Model

A fascinating and far-reaching extension concerns fruit in a supermarket. The weekly ads often give a size, like "15-size cantaloupe". This applies to any fruit that is large enough to buy individually, like grapefruit, pears, or lemons, but not to fruit like blueberries, cherries, or currants. Students could investigate what these numbers mean. Is a 12-size cantaloupe smaller or bigger than a 15-size cantaloupe? What do these numbers have to do with the way cantaloupes are packed and shipped? In fact, you could base a goodly portion of a geometry course on the desire to understand the answers to such questions.

A quick beginning of an answer is that cantaloupes, for example, most commonly come in one of the following sizes: 9, 12, 15, 18, 23, and 30. The smaller the number, the larger the cantaloupe. Why? The numbers indicate how many will fit into a standard 40-pound case or shipping box, so 9-size is the largest. The shapes of standard boxes are carefully chosen so that the right number of melons of any one size will fit comfortably but with very little wasted space into the same standard-size box. Avocado sizes come in all multiples of 4 from 20 to 40, and then 48, 60, 70, 84, and 96. The most common pear sizes are 70, 100, 150, and 215. Boxes are marked on the outside with the size numbers of the contents. If you know someone in the fruit and vegetable department of your supermarket, for example, have a look at the clever shapes of the boxes which are adaptable to contents of different sizes.

You can begin thinking about boxes for packing fruit by thinking of one layer. Then it becomes, to a reasonable first approximation, a two-dimensional problem such as finding the minimum size of a square that holds n^2 circles of radius r. What is the density of such an arrangement? This is better than any other rectangular arrangement when n is small, but eventually an arrangement more like B than A of question 10 comes to have a higher density than an arrangement within a square. Or does it? Investigate the smallest rectangular area into which to pack k circles all of radius r. Will the rectangle of smallest area that holds 7 circles in fact always hold 8?

Continue the previous investigation into three dimensions. What are different regular arrangements of spheres, and what are their densities? The problem goes back to Kepler and was first solved by Gauss. If you allow irregular packings, the problem is incredibly difficult and was finally solved only in 1998 by Thomas Hales with computer assistance. See George Szpiro's book, *Kepler's Conjecture*, for a popular account of this history.

A closely related problem is that of the so-called Kissing Number, that is, the largest number of spheres that can simultaneously touch a single sphere all of the same size. For circles in the plane the answer is 6. In three dimensions, it was the subject of a famous argument between Newton and Gregory, with a debate over whether the answer should be 12 or 13. An interesting physical experiment was done in the early 18th century. Dried peas were placed in a kettle with water and allowed to expand; the result was the peas were "formed into pretty regular Dodecahedrons" (Hales, 1731). This indicated that, perhaps, the answer should be 12, which is correct but wasn't proved until 1874!

References

Hales, S. (1731). *Statical essays: Containing vegetable staticks* (Vol. 1). London: Printed for W. Innys, T. Woodward, and J. Peele.

Szpiro, G.G. (2003). *Kepler's conjecture: How some of the greatest minds in history helped solve one of the oldest math problems in the world*. Hoboken, NJ: Wiley.